Hard-to-Solve Bimatrix Games

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Nash equilibria of bimatrix games

$$A = \begin{array}{|cc|}\hline \boxed{3} & 3 \\ \boxed{2} & 5 \\ \boxed{0} & 6 \\ \hline\end{array} \qquad\qquad B = \begin{array}{|cc|}\hline \boxed{1} & \boxed{0} \\ 0 & 2 \\ 4 & 3 \\ \hline\end{array}$$

Nash equilibrium =

pair of strategies x , y with

x best response to y and
y best response to x.

Mixed equilibria

$$A = \boxed{\begin{matrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{matrix}} \qquad B = \boxed{\begin{matrix} 1 & 0 \\ 0 & 2 \\ 4 & 3 \end{matrix}}$$

$$x = \boxed{\begin{matrix} 0 \\ 1/3 \\ 2/3 \end{matrix}} \qquad y^T = \boxed{\begin{matrix} 1/3 & 2/3 \end{matrix}}$$

$$Ay = \boxed{\begin{matrix} 3 \\ 4 \\ 4 \end{matrix}} \qquad x^T B = \boxed{\begin{matrix} 8/3 & 8/3 \end{matrix}}$$

only **pure best responses** can have probability > 0

Supports that don't work

$$A = \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 4 & 3 \end{bmatrix}$$

$$x = \begin{bmatrix} ? \\ 0 \\ ? \end{bmatrix} \qquad y^T = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$$

$$Ay = \begin{bmatrix} 3 \\ 3.5 \\ 3 \end{bmatrix}$$

infeasible for player **I**

Best response polyhedron H_2 for player 2

$$\begin{array}{c|cc|} & \overline{y}_4 & \overline{y}_5 \\ \hline ① & 3 & 3 \\ ② & 2 & 5 \\ ③ & 0 & 6 \\ \hline \end{array} = A$$

$$H_2 = \{ \ (\overline{y}_4, \overline{y}_5, u) \ |$$

① : $3\overline{y}_4 + 3\overline{y}_5 \leq u$

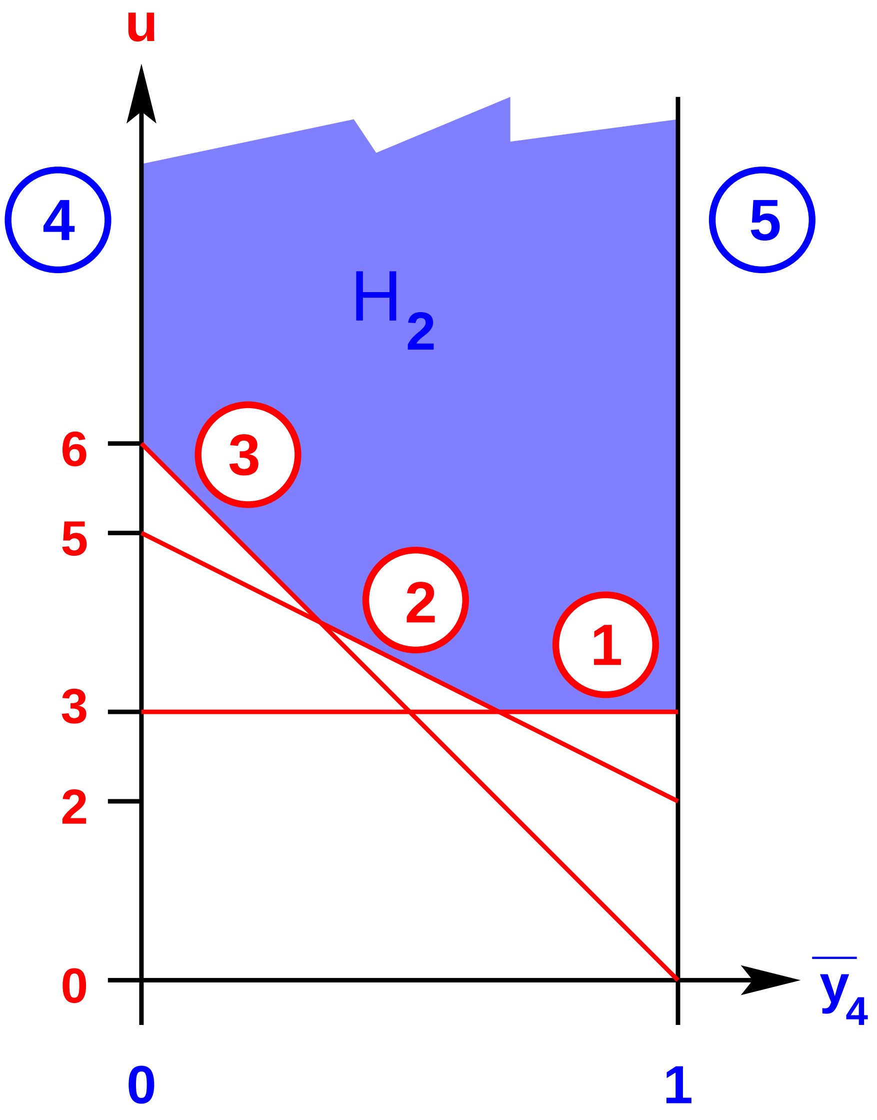
② : $2\overline{y}_4 + 5\overline{y}_5 \leq u$

③ : $6\overline{y}_5 \leq u$

$$\overline{y}_4 + \overline{y}_5 = 1$$

④ : $\overline{y}_4 \geq 0$

⑤ : $\overline{y}_5 \geq 0 \ \}$



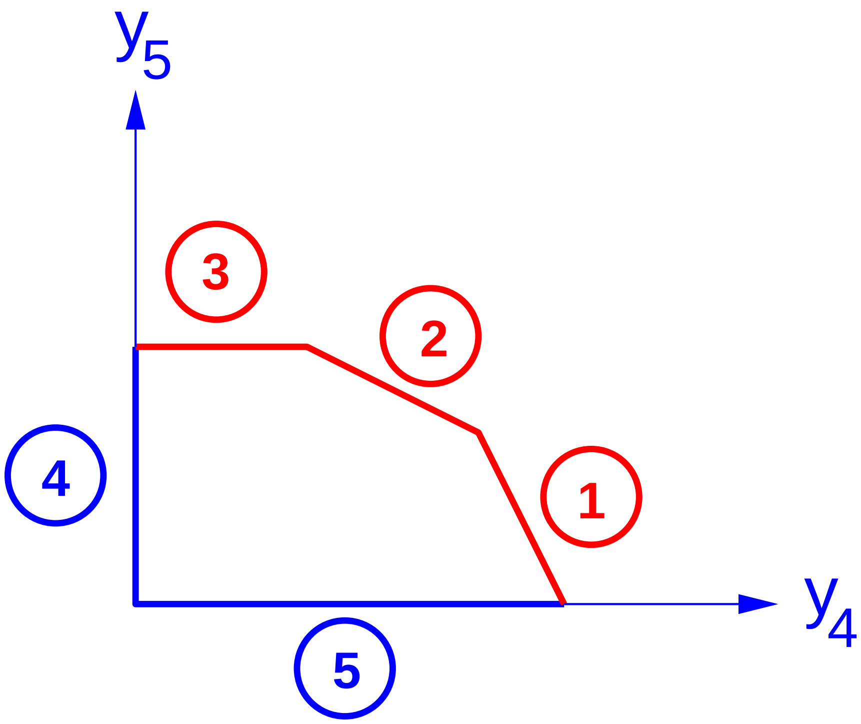
Best response polytope Q for player 2

$$\begin{array}{c|cc|} & y_4 & y_5 \\ \hline \textcircled{1} & 3 & 3 \\ \textcircled{2} & 2 & 5 \\ \textcircled{3} & 0 & 6 \\ \hline \end{array} = A$$

$$Q = \{\, y \mid Ay \leq 1,\ y \geq 0 \,\}$$

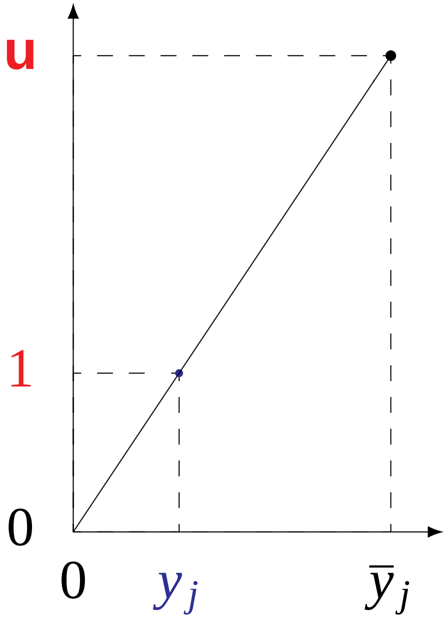
$$Q = \{\ (y_4, y_5)\ \mid$$

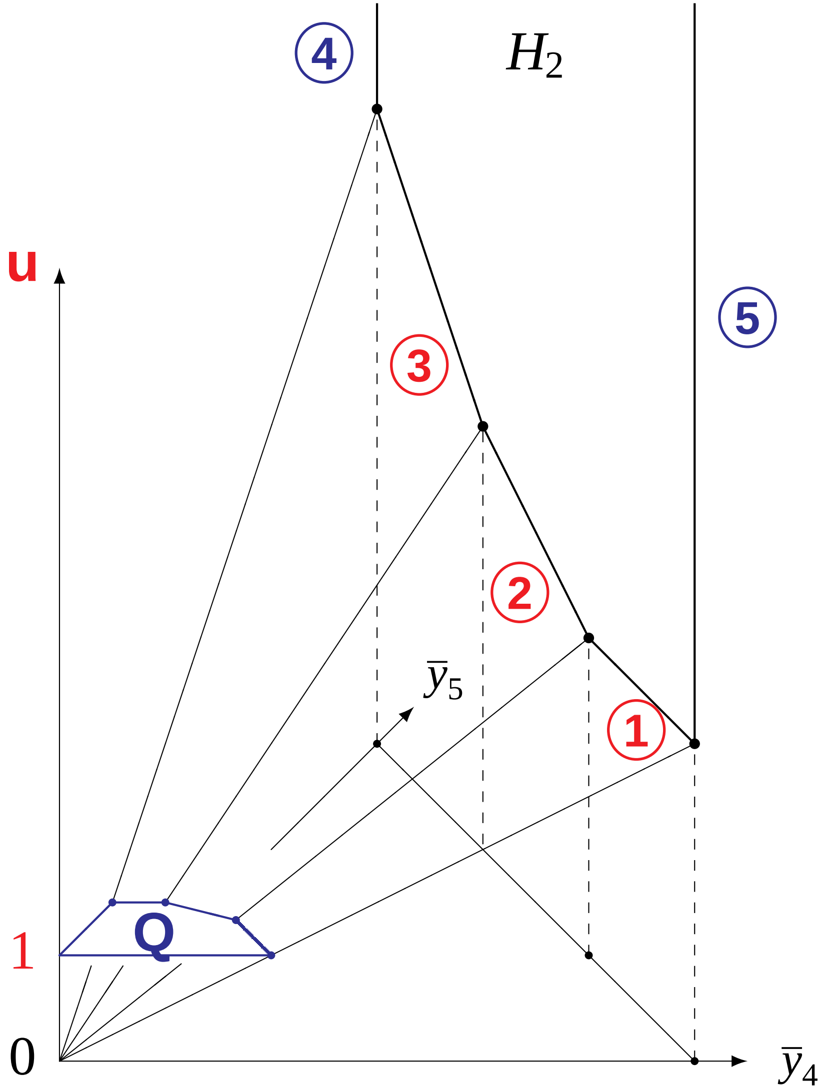
$$\begin{array}{rl} \textcircled{1}: & 3y_4 + 3y_5 \leq 1 \\ \textcircled{2}: & 2y_4 + 5y_5 \leq 1 \\ \textcircled{3}: & 6y_5 \leq 1 \\ & \\ \textcircled{4}: & y_4 \geq 0 \\ \textcircled{5}: & y_5 \geq 0 \ \} \end{array}$$



Projective transformation

H_2, **Q** same face incidences





Best response polytope Q for player 2

$$\begin{array}{c|cc|} & y_4 & y_5 \\ \hline ① & 3 & 3 \\ ② & 2 & 5 \\ ③ & 0 & 6 \\ \hline \end{array} = A$$

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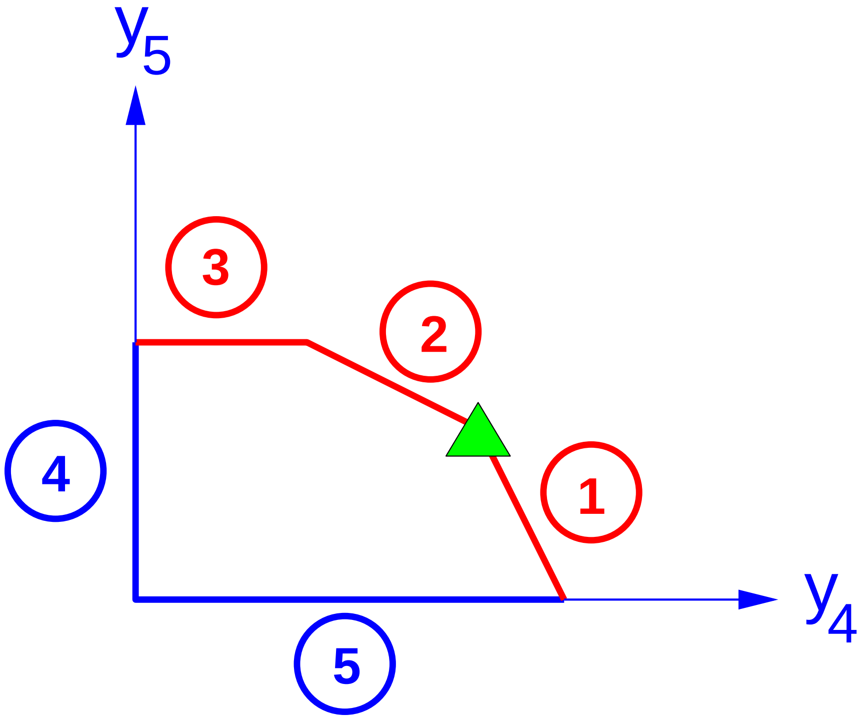
$$Q = \{\ (y_4, y_5)\ \mid$$

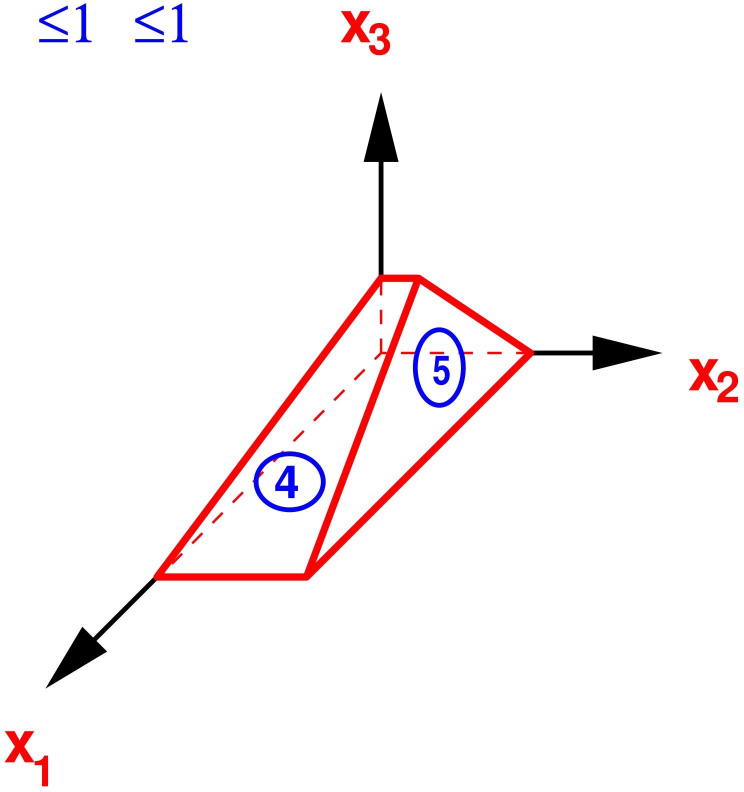
$$\begin{array}{rl} ①: & 3y_4 + 3y_5 \leq 1 \\ ②: & 2y_4 + 5y_5 \leq 1 \\ ③: & 6y_5 \leq 1 \\ & \\ ④: & y_4 \geq 0 \\ ⑤: & y_5 \geq 0 \ \} \end{array}$$

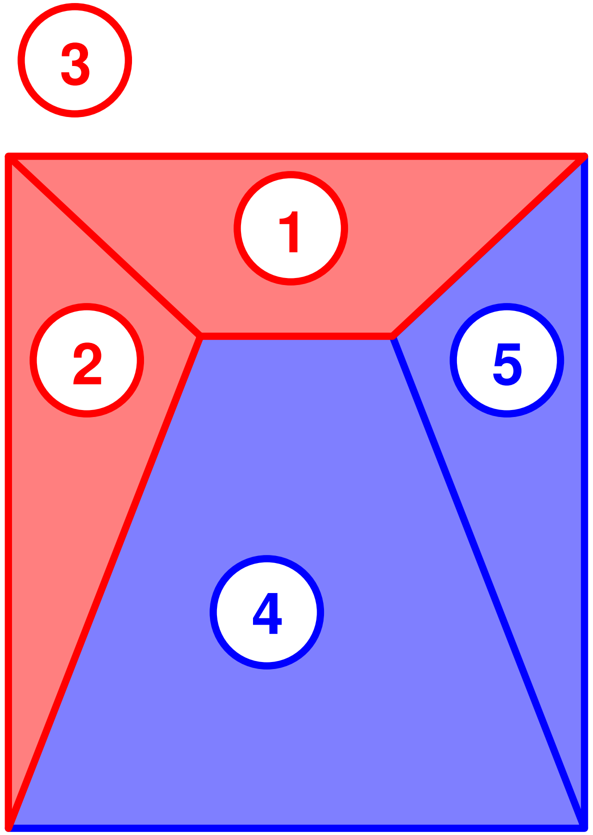


Best response polytope P for player 1

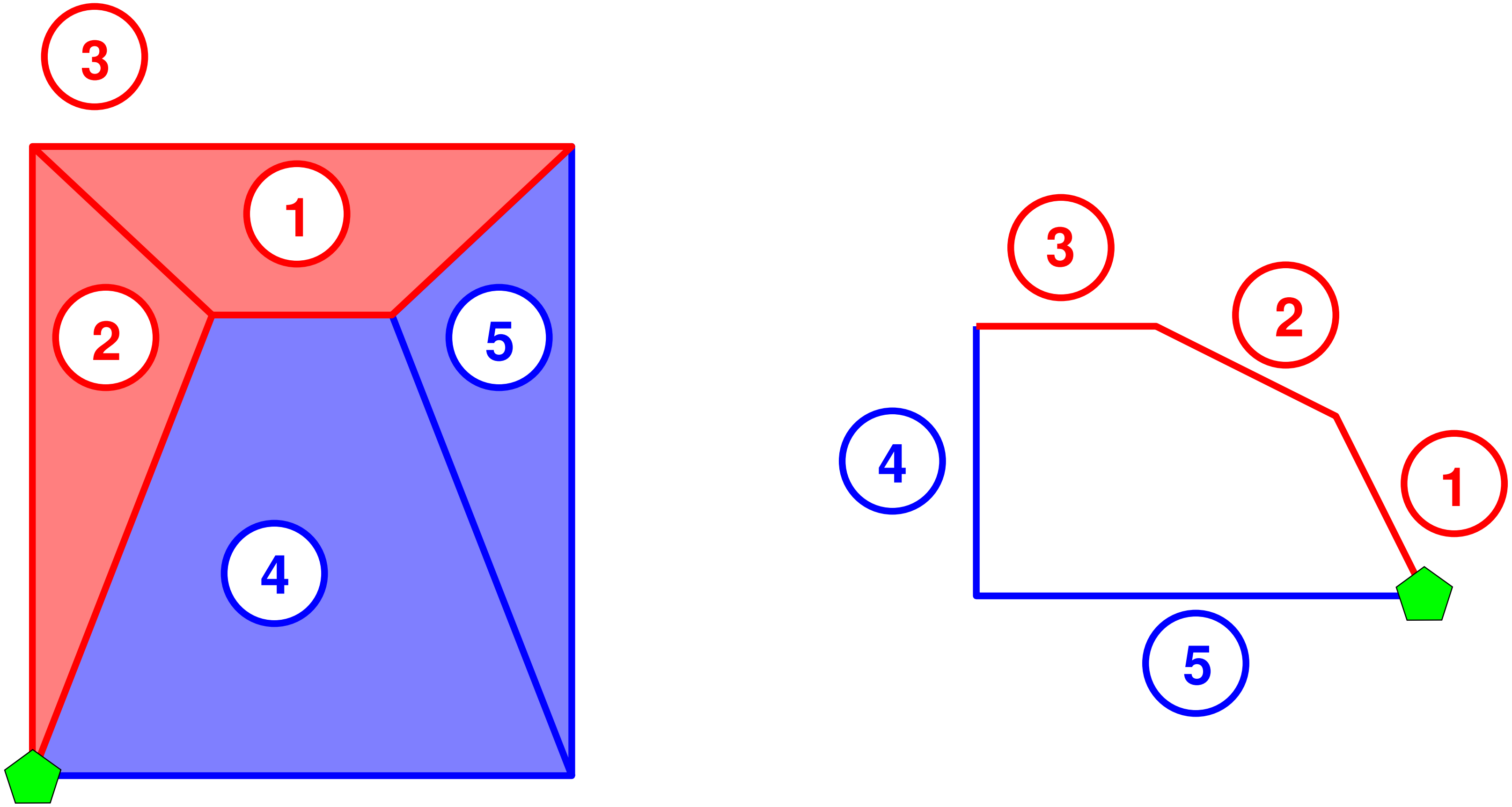
$$\begin{array}{c|cc|} x_1 & 1 & 0 \\ x_2 & 0 & 2 \\ x_3 & 4 & 3 \\ \hline & \le 1 & \le 1 \end{array} = B$$

$$P = \{ x \mid x \ge 0, \; x^\top B \le 1 \}$$



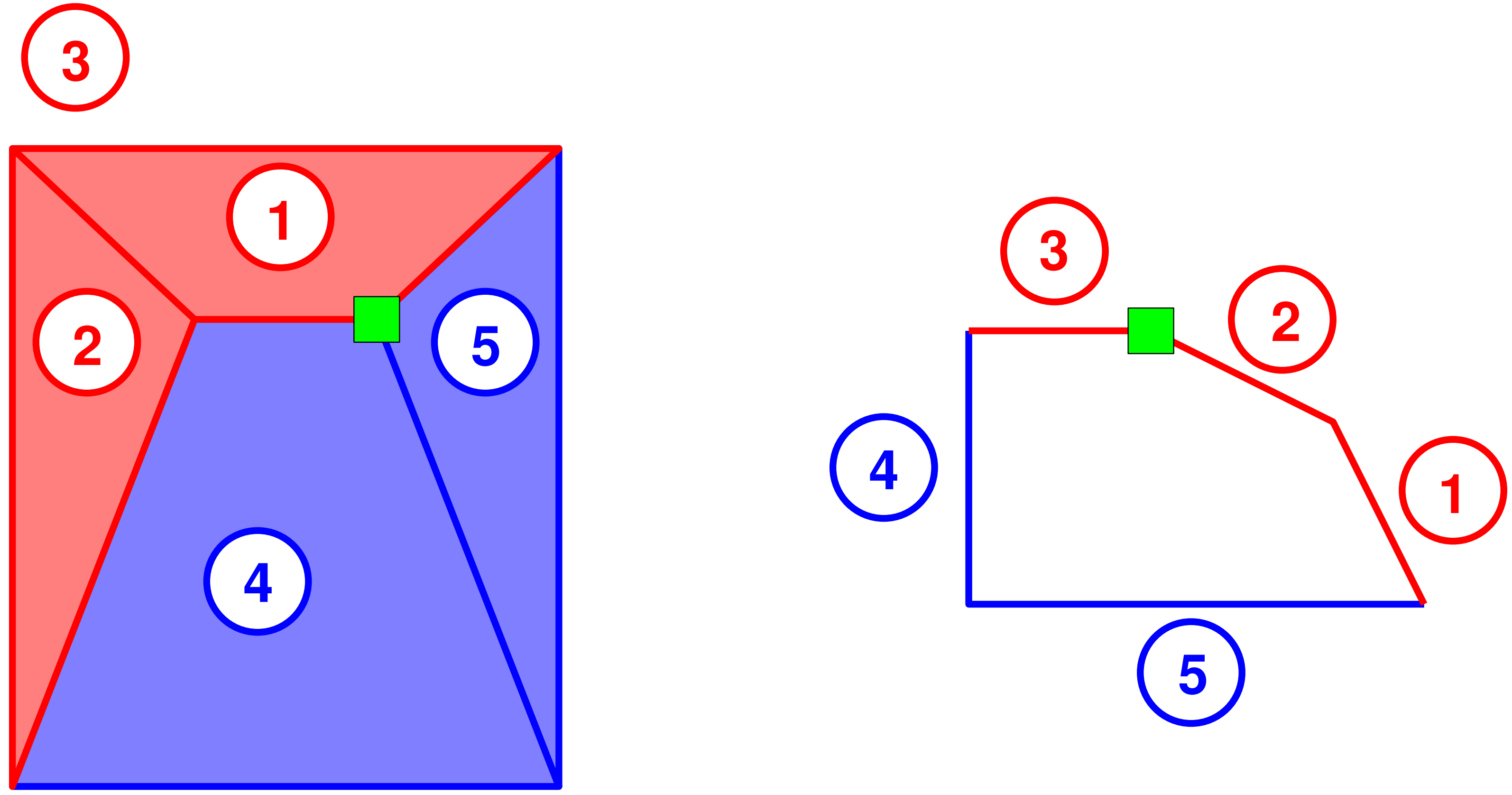


Equilibrium = completely labeled pair



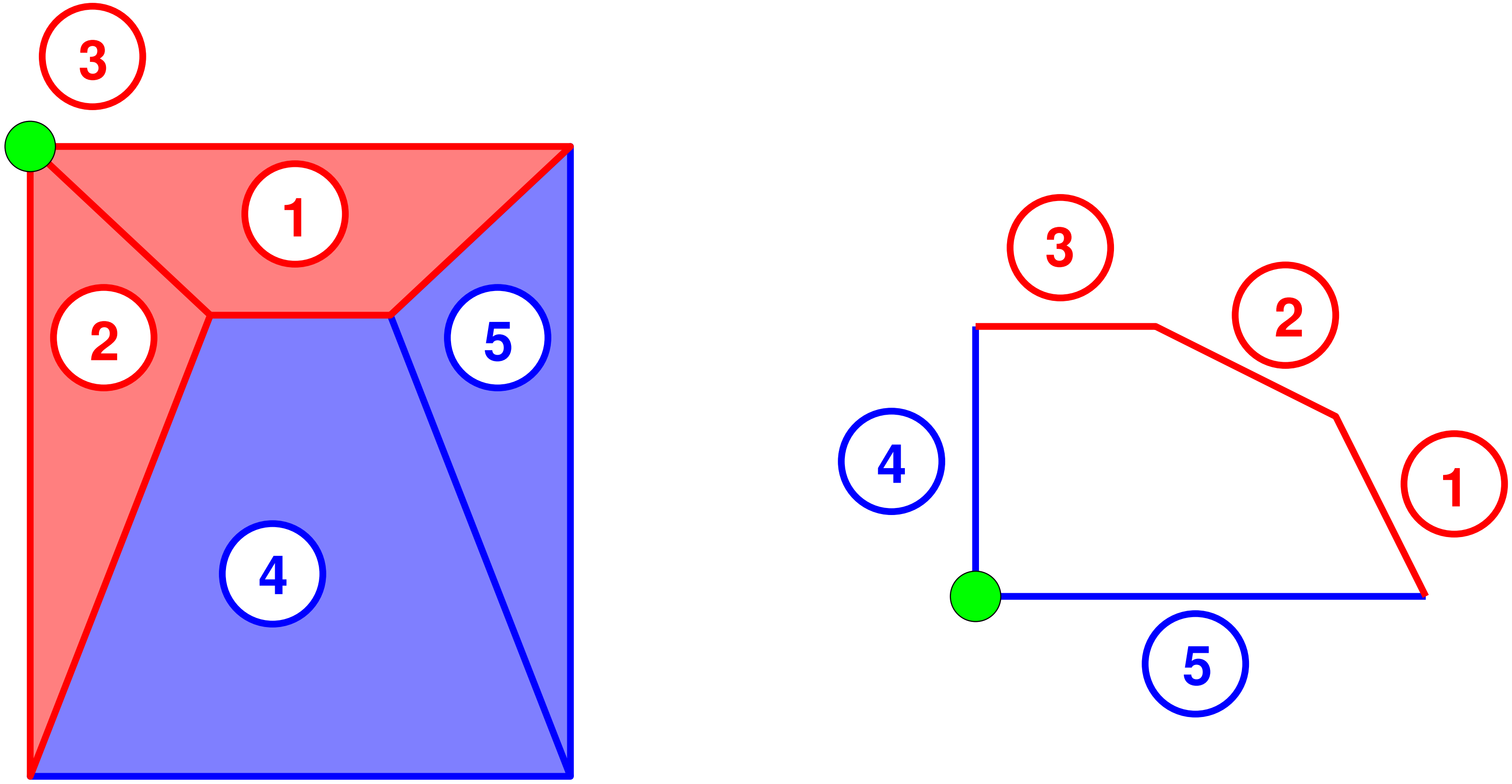
pure equilibrium

Equilibrium = completely labeled pair

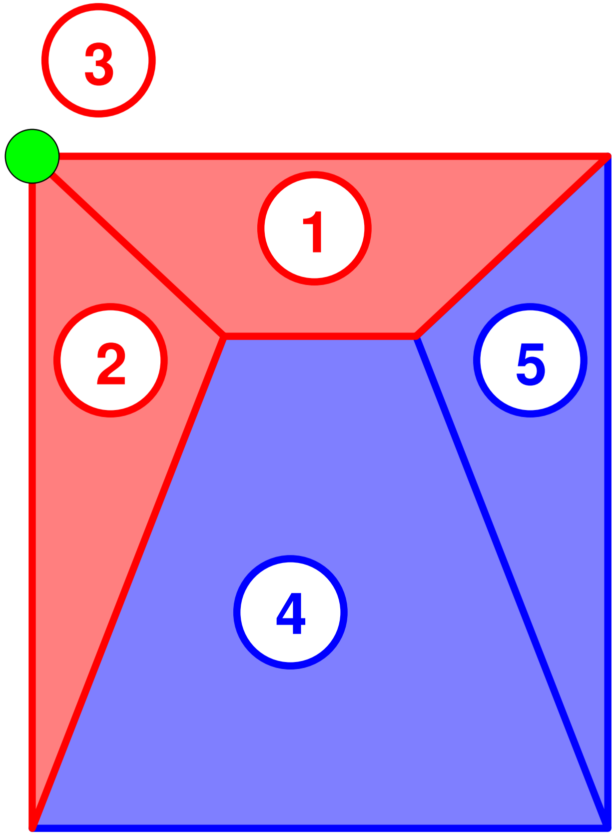


mixed equilibrium

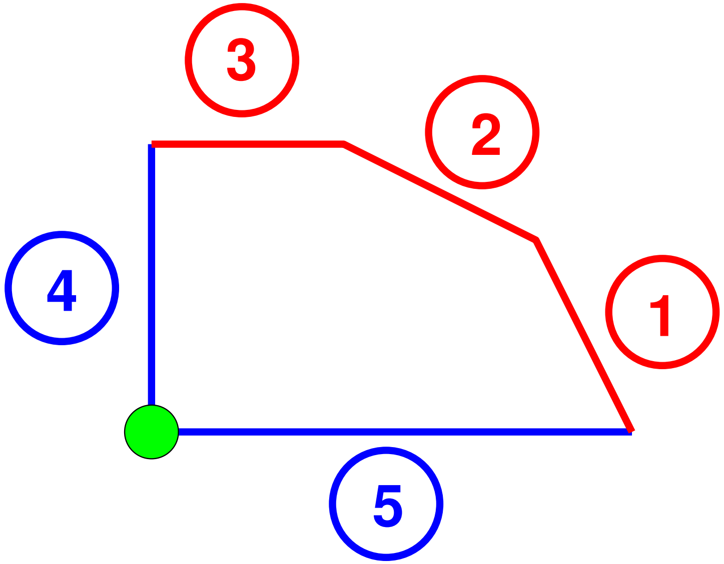
The Lemke–Howson algorithm



The Lemke–Howson algorithm



Drop label ③



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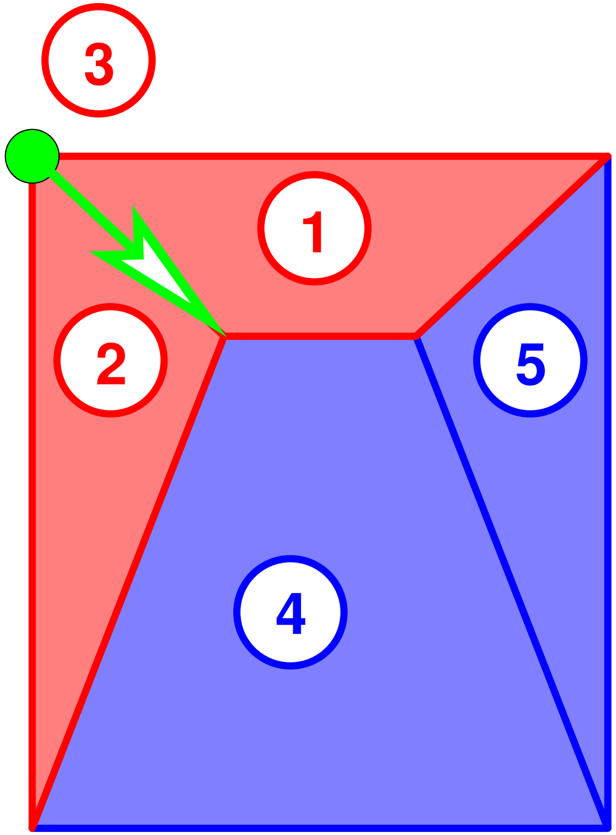

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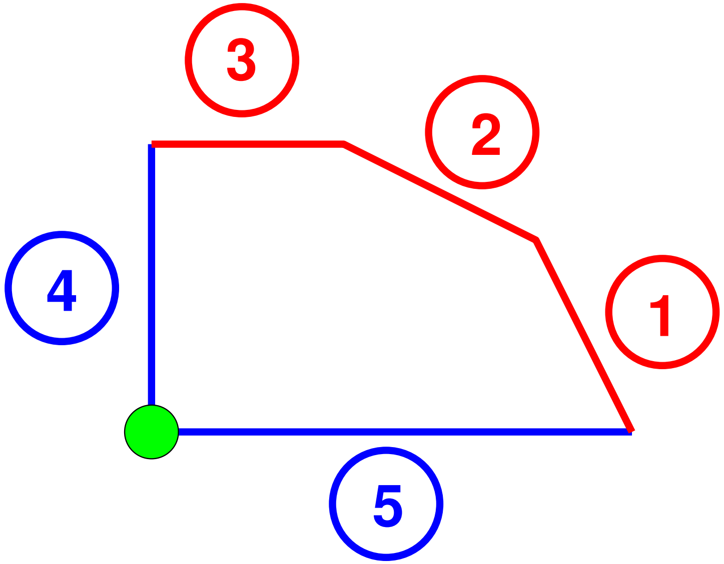

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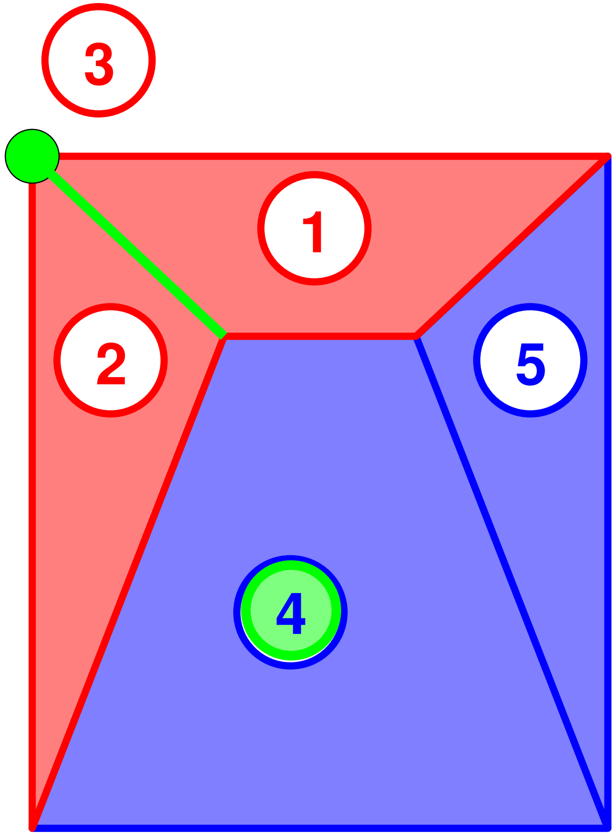
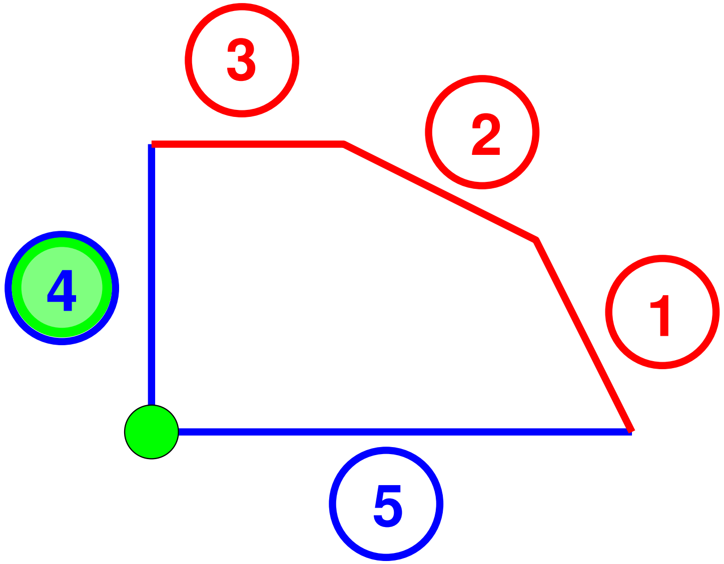


Drop label 3



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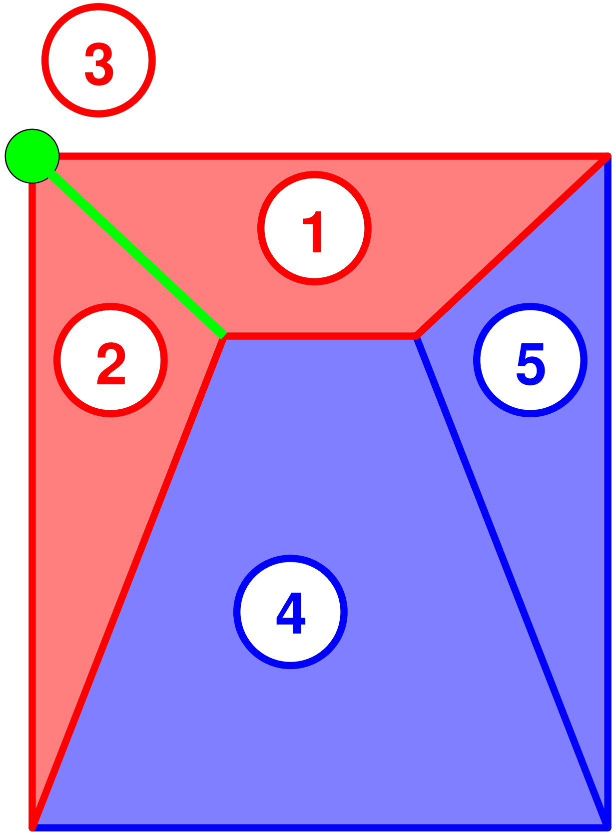

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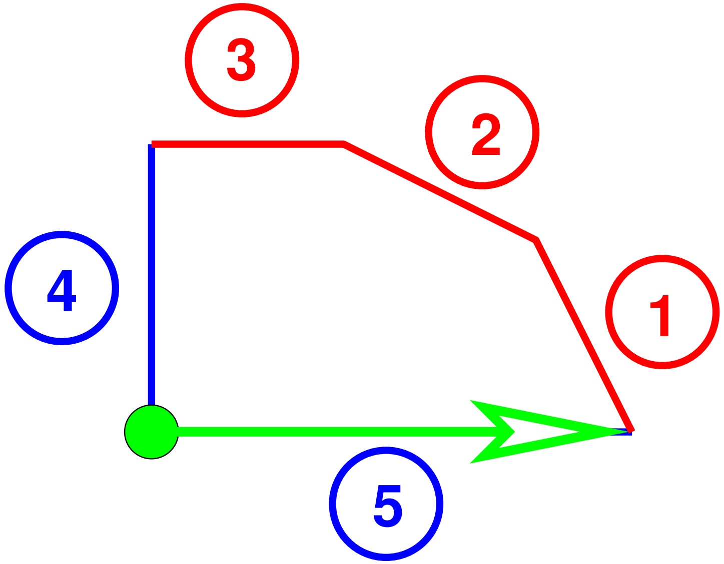

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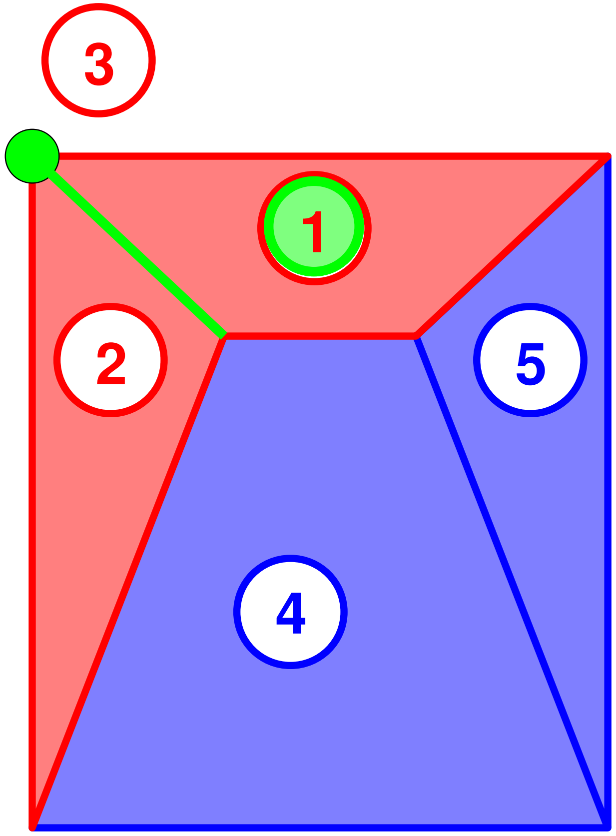

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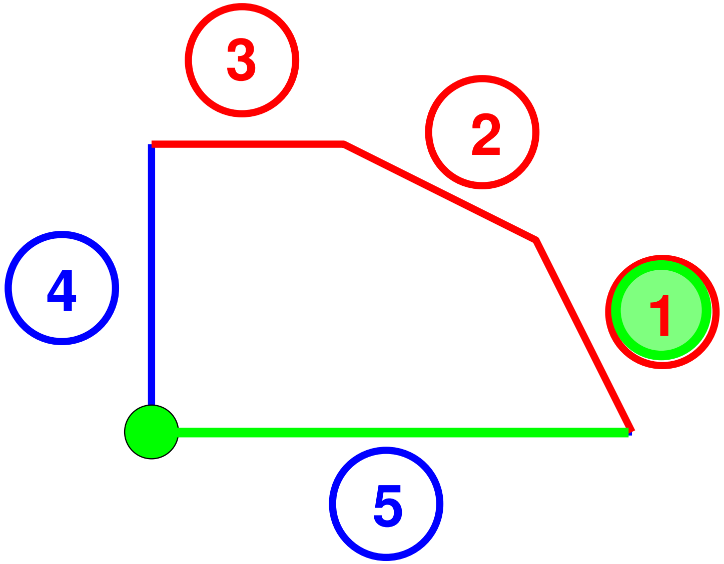

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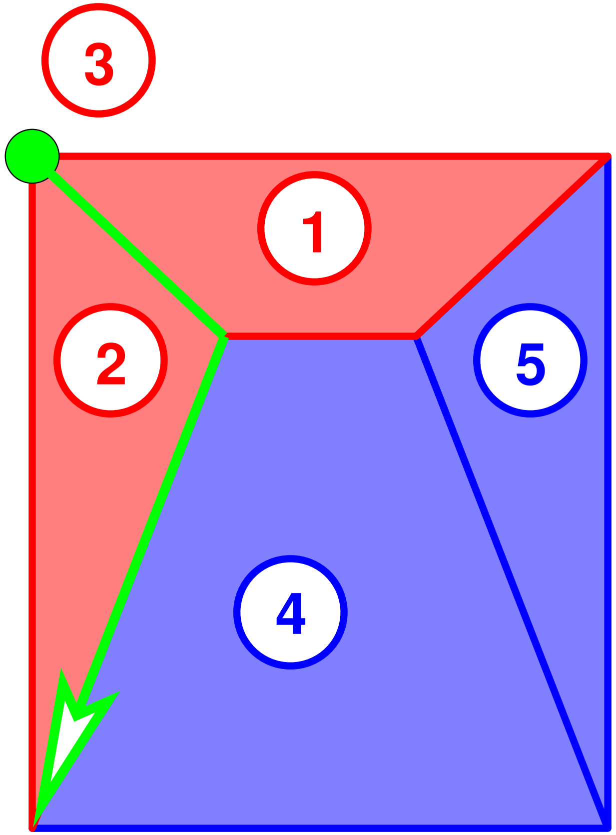

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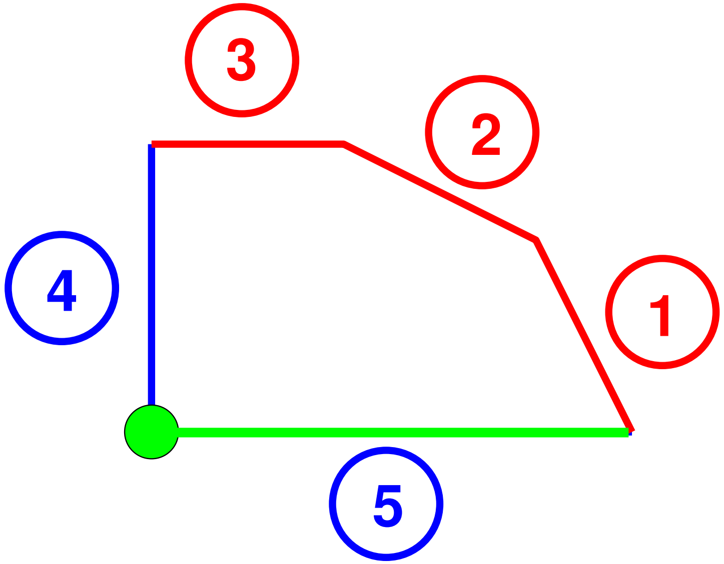

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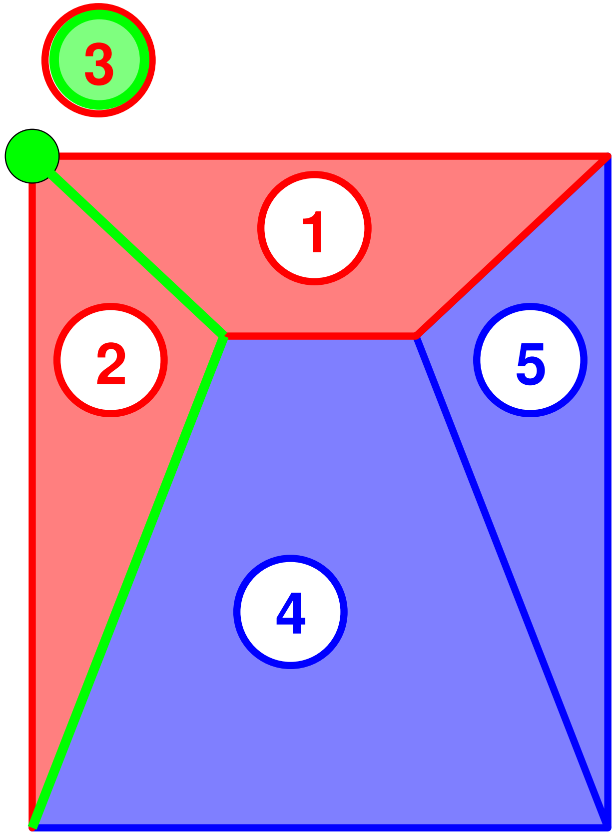

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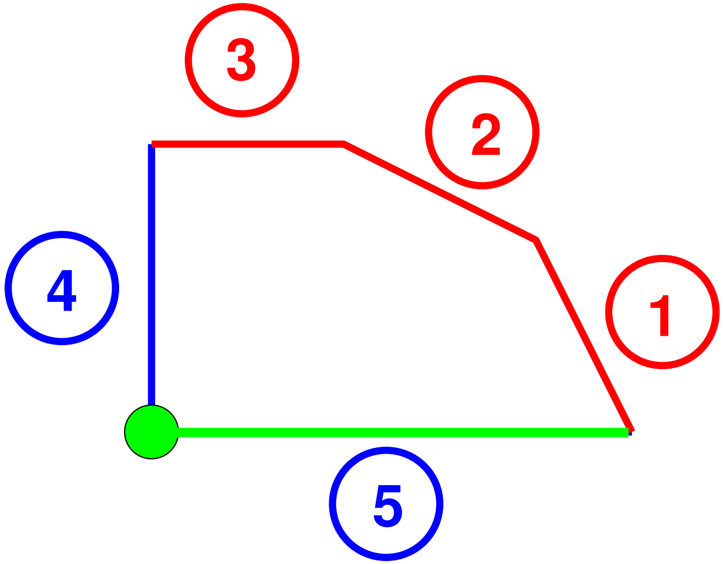

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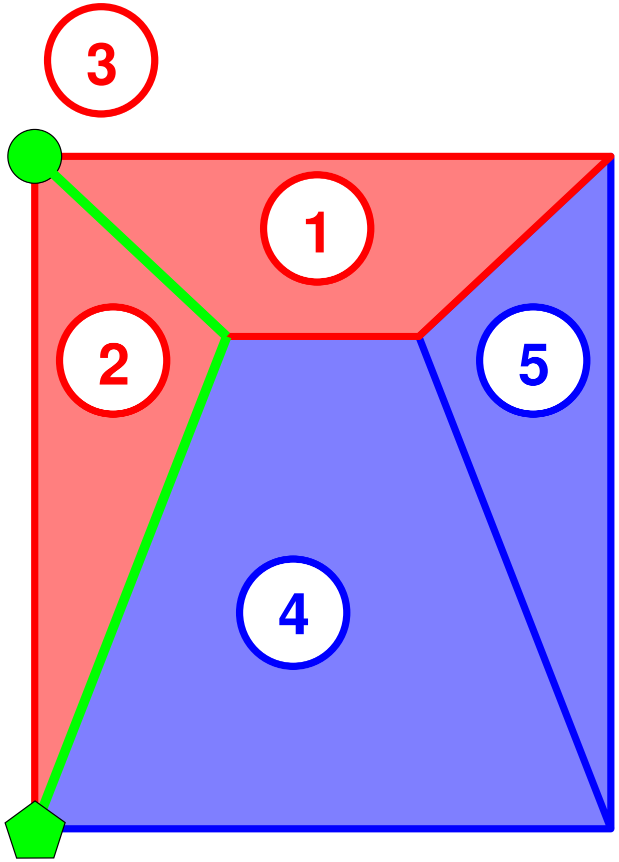

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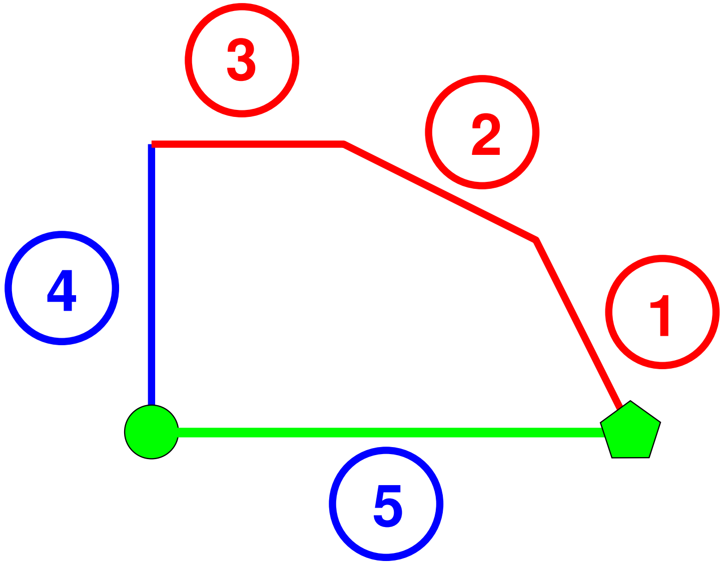


Why Lemke-Howson works

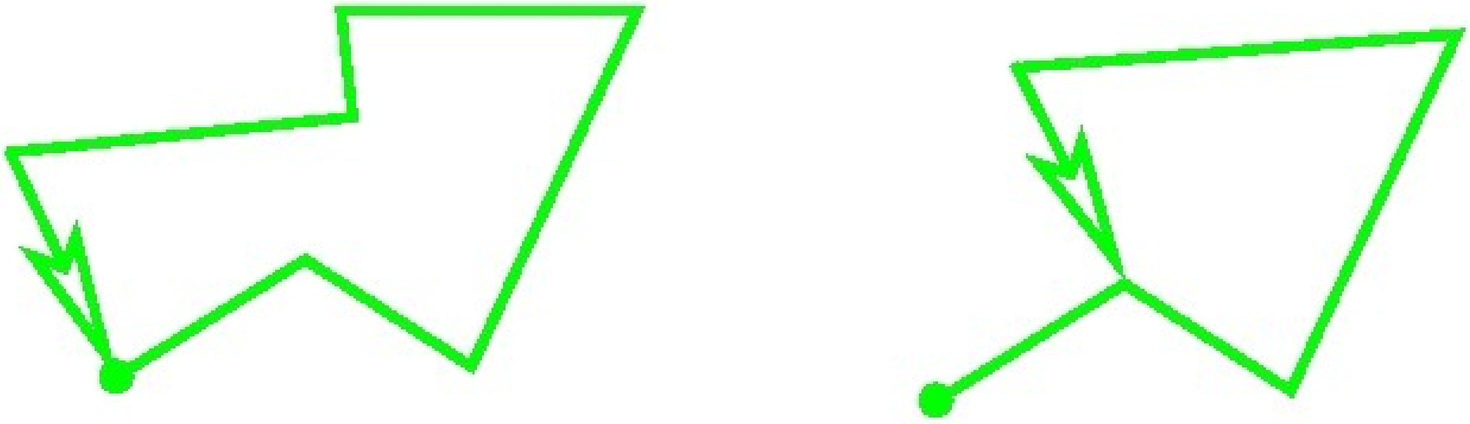
LH finds at least one Nash equilibrium because

- finitely many "vertices"

for nondegenerate (generic) games:

- **unique** starting edge given missing label
- **unique** continuation

⇒ precludes "coming back" like here:



Complexity of Lemke-Howson

- finds at least one Nash equilibrium,
  pivots like Simplex algorithm for linear programming

- Simplex may be exponential [Klee-Minty cubes]

- exponentially many steps of Lemke-Howson
  for **any** dropped label?

- **Yes**! This is our result.

Our result

There are $d \times d$ games with exactly one Nash equilibrium, for which the Lemke-Howson algorithm takes $\geq \phi^{3d/4}$ many steps for **any dropped label** (with Golden Ratio $\phi = (\sqrt{5} + 1) / 2 = 1.618...$)

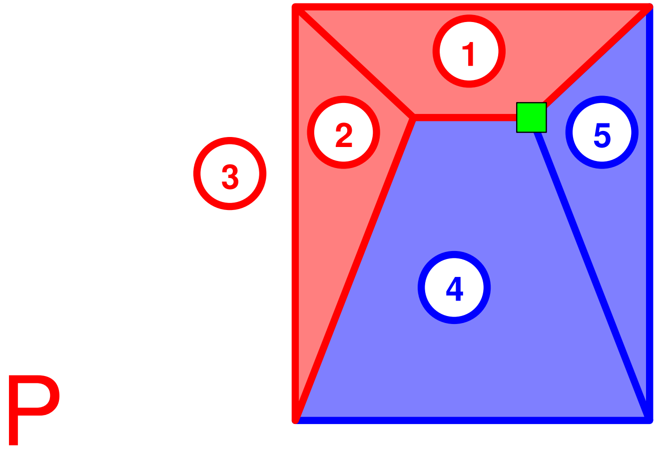
We will show this extending

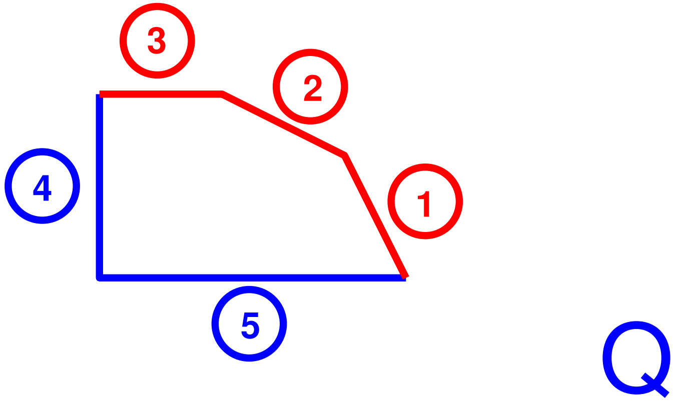
[Morris 1994] - exponentially long Lemke paths
(finds symmetric equilibria of symmetric games)

[von Stengel 1999] - games with many equilibria

using **dual cyclic polytopes**

Vertices as bit patterns

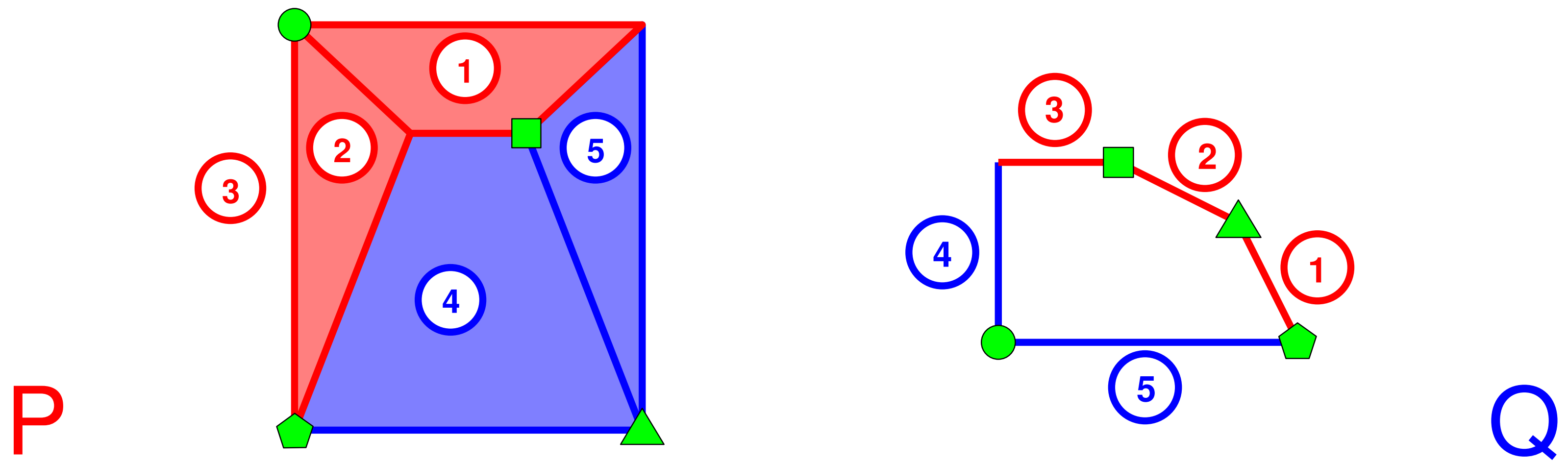


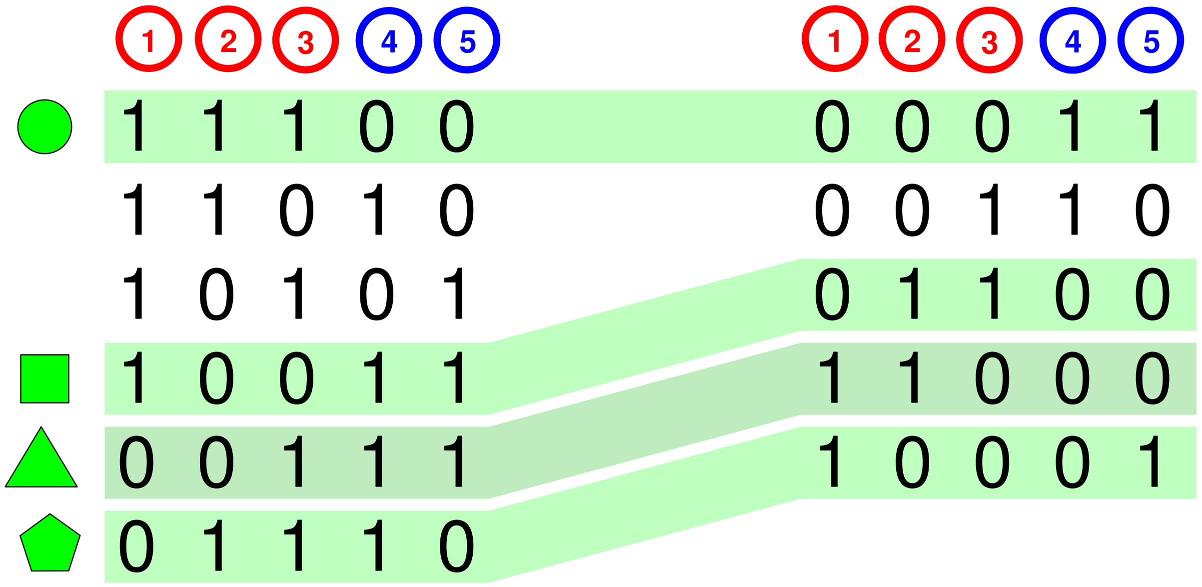


	①	②	③	④	⑤
	1	1	1	0	0
	1	1	0	1	0
	1	0	1	0	1
🟩	1	0	0	1	1
	0	0	1	1	1
	0	1	1	1	0

①	②	③	④	⑤
0	0	0	1	1
0	0	1	1	0
0	1	1	0	0
1	1	0	0	0
1	0	0	0	1

Vertices as bit patterns





	①	②	③	④	⑤	①	②	③	④	⑤
●	1	1	1	0	0	0	0	0	1	1
	1	1	0	1	0	0	0	1	1	0
	1	0	1	0	1	0	1	1	0	0
■	1	0	0	1	1	1	1	0	0	0
▲	0	0	1	1	1	1	0	0	0	1
⬟	0	1	1	1	0					

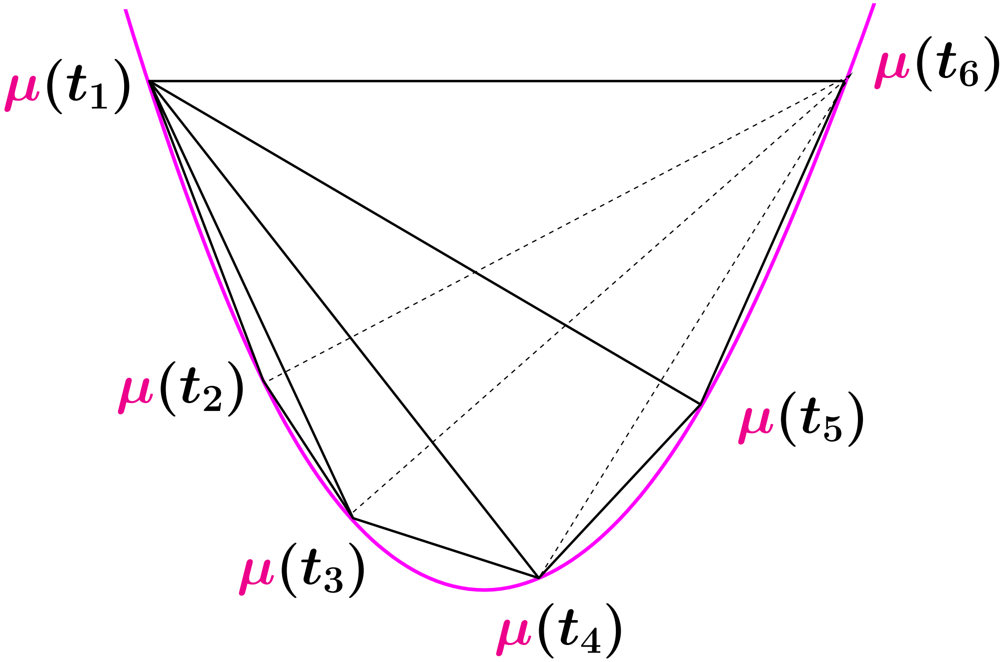
Cyclic polytopes

moment curve in $\mathbf{R}^d$

$\mu : \mathbf{R} \longrightarrow \mathbf{R}^d \qquad t \mapsto \mu(t) = (t, t^2, \ldots, t^d)^\top.$

cyclic polytope in dim d with N vertices: $\quad t_1 < t_2 < \cdots < t_N$

$$C_d(N) := \mathrm{conv}\{\mu(t_1), \ldots, \mu(t_N)\}$$

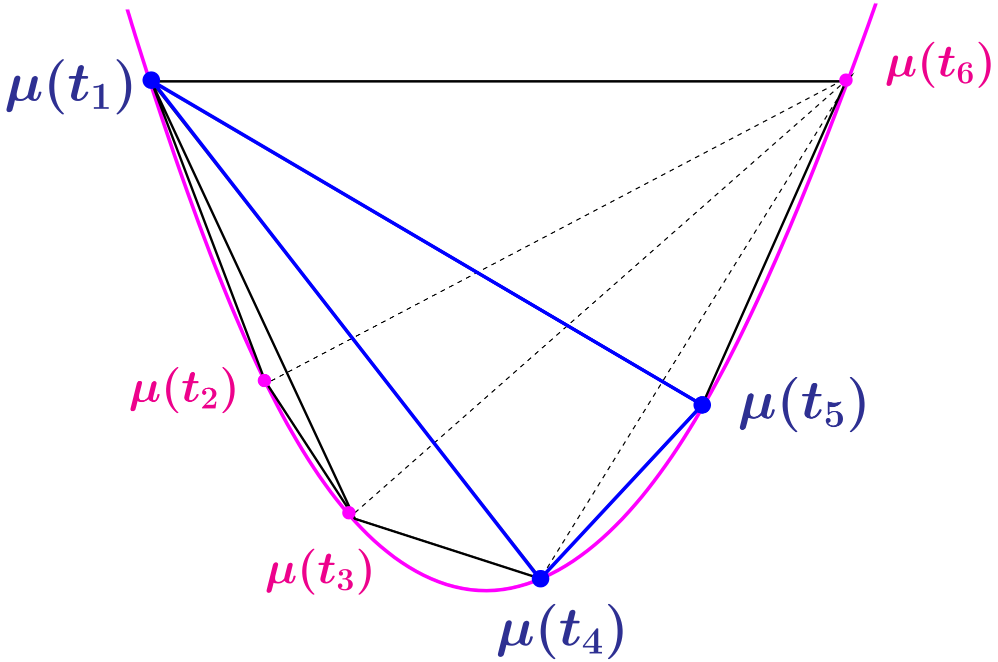


Facets of $C_d(N)$

Any d of the vertices $\mu(t_1), \ldots, \mu(t_N)$ define hyperplane F in $\mathbf{R}^d$.

F facet $\iff$ all **other** vertices are on one side of F

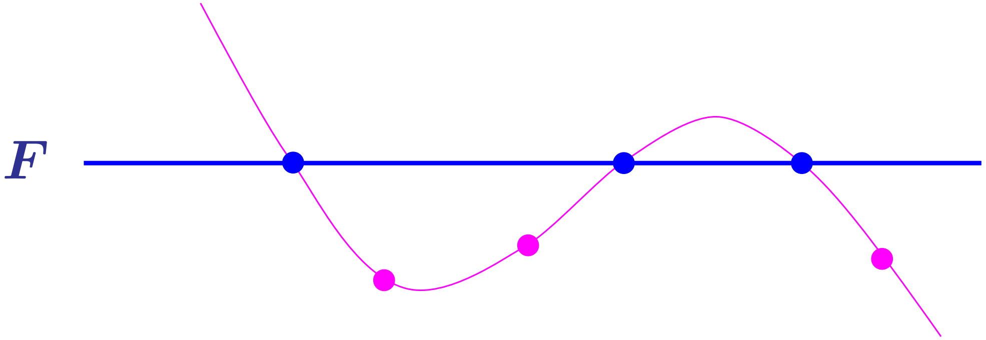
Example: $C_3(6)$, vertices 100110



Gale's Evenness condition

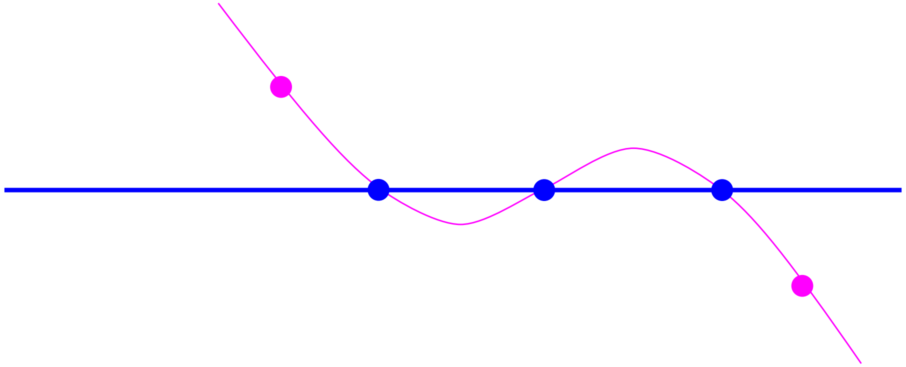
bitstring $s = \; s_1 s_2 \ldots s_N$, $\qquad s_i \in \{0, 1\}$ $\qquad$ e.g. 100110

defines facet F = conv$\{\mu(t_i) \mid s_i = 1\}$ of $C_d(N)$



$\Longleftrightarrow$ s has only even-length substrings 0110, 011110, 01111110,

forbidden: substrings 010, 01110, $\ldots$ of odd length.

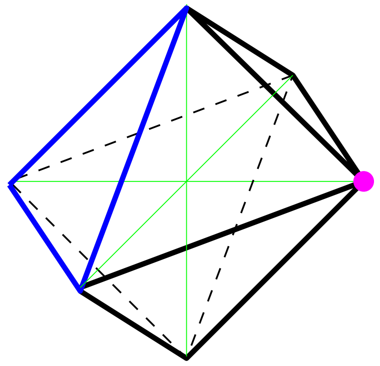


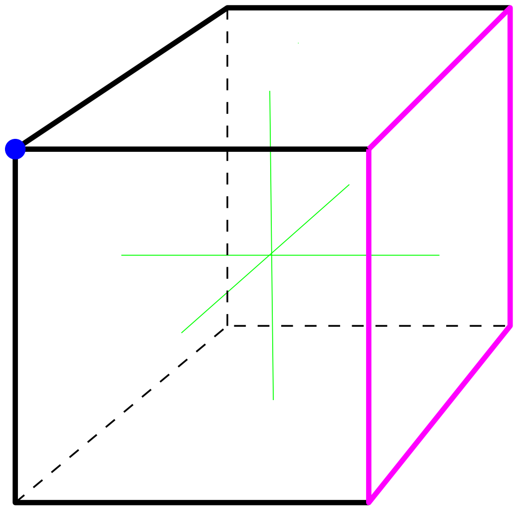
Polar polytopes

P = conv$\{c_1, \dots, c_N\}$, $0 \in \operatorname{int}(P)$ vertices c_i

polar polytope

$P^\Delta = \{\, z \mid {c_1}^\top z \le 1, \dots, \ {c_N}^\top z \le 1 \}$ facets $\{z \in P^\Delta \mid {c_i}^\top z = 1\}$





Dual cyclic polytopes

- vertices = strings of **N** bits with **d** bits "**1**",
- **no** odd substrings 010, 01110, 0111110, . . .
  [Gale **even**ness]

Example: d=4, N=6 d=2, N=6 (4 × 2 game)

d=4, N=6

111100
111001
110110
110011
101101
100111
011110
011011
001111

d=2, N=6

000011
000110
001100
011000
110000
100001

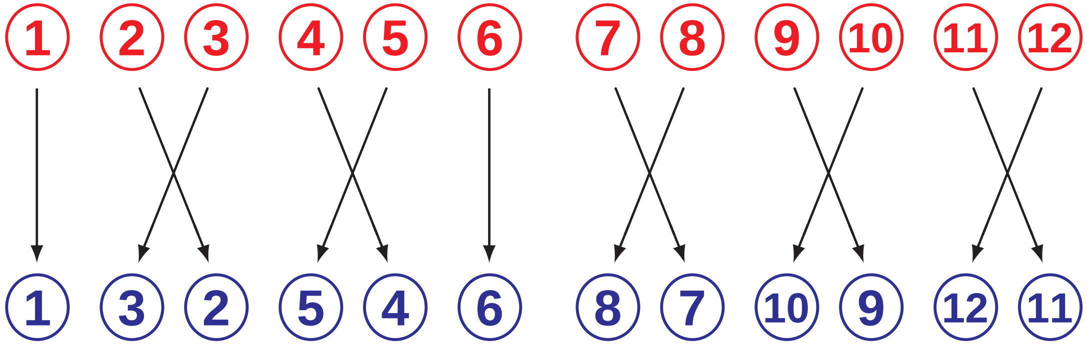
Vertices of $C_d(2d)^\Delta$ and complementarity

$C_4(8)^\Delta$

vertex no.	defining facets	labels (example)
1	00001111	
2	00011011	
3	00011110	
4	00110011	
5	00110110	
6	00111100	
7	01100011	
8	01100110	② ③ ⑥ ⑦
9	01101100	
10	01111000	
11	10000111	
12	10001101	
13	10011001	① ④ ⑤ ⑧
14	10110001	
15	11000011	
16	11000110	
17	11001100	
18	11011000	
19	11100001	
20	11110000	

Permuted labels

P = dual cyclic polytope in dimension **d** with **2d** facets



with facets labeled ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨ ⑩ ⑪ ⑫

Q = P

with facets labeled ① ③ ② ⑤ ④ ⑥ ⑧ ⑦ ⑩ ⑨ ⑫ ⑪

⟹ only **one** non-artificial equilibrium:

0 0 0 0 0 0 1 1 1 1 1 1

1 1 1 1 1 1 0 0 0 0 0 0

Lemke–Howson will take long to find it!

Lemke-Howson on dual cyclic polytopes

P								Q							
①	②	③	④	⑤	⑥	⑦	⑧	①	③	②	④	⑥	⑤	⑧	⑦
$\boxed{1}$	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1

Lemke-Howson on dual cyclic polytopes

P								Q							
①	②	③	④	⑤	⑥	⑦	⑧	①	③	②	④	⑥	⑤	⑧	⑦
1	1	1	1	0	0	0	0	0	0	0	0	1	$\boxed{1}$	1	1
0	1	1	1	1	0	0	0								

Lemke-Howson on dual cyclic polytopes

P								**Q**							
①	②	③	④	⑤	⑥	⑦	⑧	①	③	②	④	⑥	⑤	⑧	⑦
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
0	1	1	$\boxed{1}$	1	0	0	0	0	0	0	1	1	0	1	1

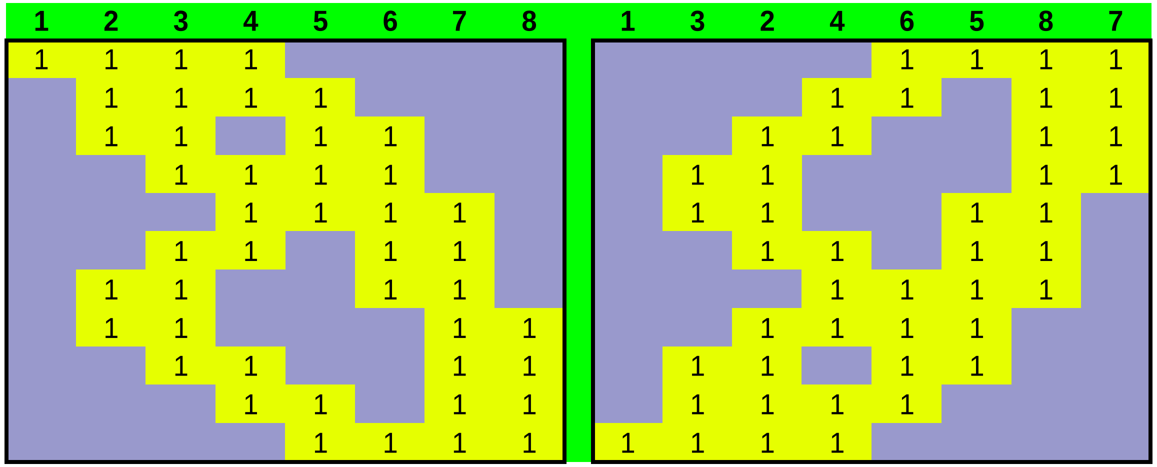
Lemke-Howson on dual cyclic polytopes

P								**Q**							
①	②	③	④	⑤	⑥	⑦	⑧	①	③	②	④	⑥	⑤	⑧	⑦
1	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**
0	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	$\boxed{1}$	**0**	**1**	**1**
0	**1**	**1**	**0**	**1**	**1**	**0**	**0**								

Lemke-Howson on dual cyclic polytopes

P								Q							
①	②	③	④	⑤	⑥	⑦	⑧	①	③	②	④	⑥	⑤	⑧	⑦
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
0	1	1	1	1	0	0	0	0	0	0	1	1	0	1	1
0	1	1	0	1	1	0	0	0	0	1	1	0	0	1	1
0	0	1	1	1	1	0	0	0	1	1	0	0	0	1	1
0	0	0	1	1	1	1	0	0	1	1	0	0	1	1	0
0	0	1	1	0	1	1	0	0	0	1	1	0	1	1	0
0	1	1	0	0	1	1	0	0	0	0	1	1	1	1	0
0	1	1	0	0	0	1	1	0	0	1	1	1	1	0	0
0	0	1	1	0	0	1	1	0	1	1	0	1	1	0	0
0	0	0	1	1	0	1	1	0	1	1	1	1	0	0	0
0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0

A(4) = path for d=4, label 1



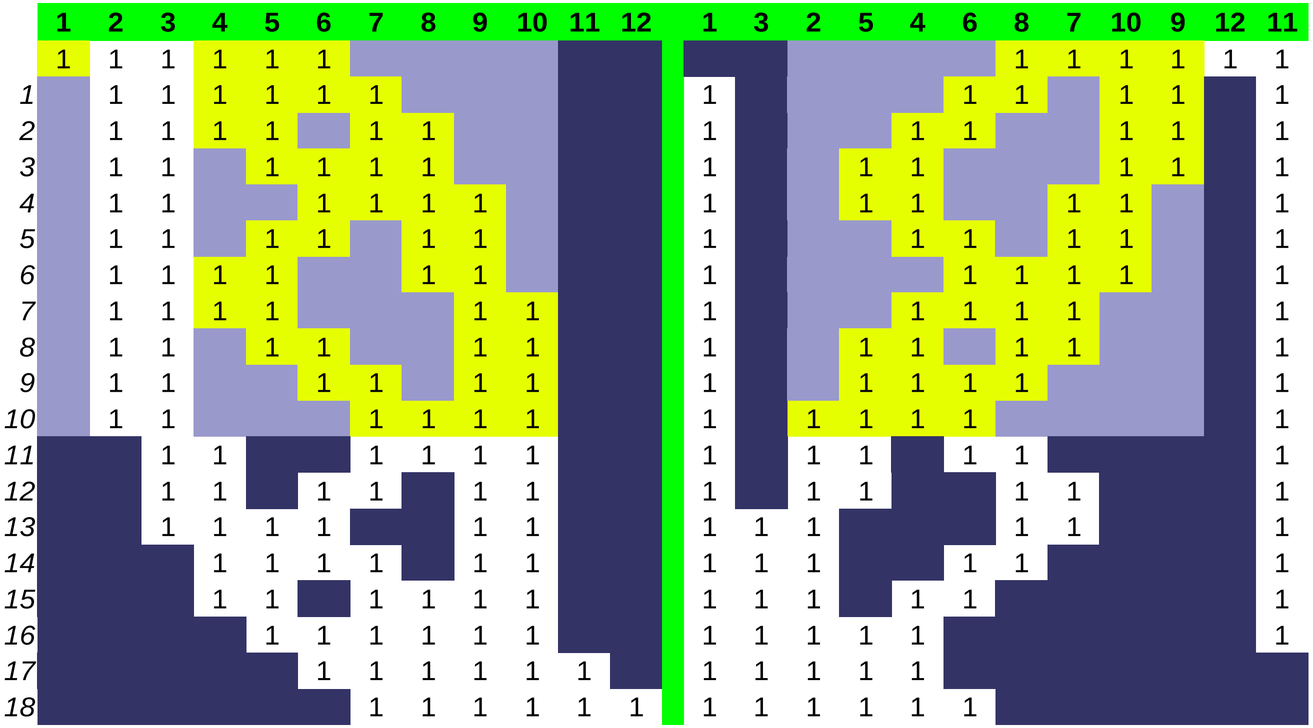
1	2	3	4	5	6	7	8
1	1	1	1				
	1	1	1	1			
	1	1		1	1		
		1	1	1	1		
			1	1	1	1	
		1	1		1	1	
	1	1			1	1	
	1	1				1	1
		1	1			1	1
			1	1		1	1
				1	1	1	1

1	3	2	4	6	5	8	7
				1	1	1	1
			1	1		1	1
		1	1			1	1
	1	1				1	1
	1	1			1	1	
		1	1		1	1	
			1	1	1	1	
		1	1	1	1		
	1	1		1	1		
	1	1	1	1			
1	1	1	1				

B(6) = path for d=6, label 12

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**		**1**	**3**	**2**	**5**	**4**	**6**	**8**	**7**	**10**	**9**	**12**	**11**
	1	1	1	1	1	1														1	1	1	1	1	1
1		1	1	1	1	1	1							1					1	1		1	1		1
2		1	1	1	1		1	1						1				1	1			1	1		1
3		1	1		1	1	1	1						1			1	1				1	1		1
4		1	1			1	1	1	1					1			1	1			1	1			1
5		1	1		1	1		1	1					1				1	1		1	1			1
6		1	1	1	1			1	1					1					1	1	1	1			1
7		1	1	1	1				1	1				1				1	1	1	1				1
8		1	1		1	1			1	1				1			1	1		1	1				1
9		1	1			1	1		1	1				1			1	1	1	1					1
10		1	1				1	1	1	1				1		1	1	1	1						1
11			1	1			1	1	1	1				1		1	1		1	1					1
12			1	1		1	1		1	1				1		1	1			1	1				1
13			1	1	1	1			1	1				1	1	1				1	1				1
14				1	1	1	1		1	1				1	1	1			1	1					1
15				1	1		1	1	1	1				1	1	1		1	1						1
16					1	1	1	1	1	1				1	1	1	1	1							1
17						1	1	1	1	1	1			1	1	1	1	1							
18							1	1	1	1	1	1		1	1	1	1	1	1						

A(4) is prefix of B(6)



	1	2	3	4	5	6	7	8	9	10	11	12		1	3	2	5	4	6	8	7	10	9	12	11
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1		1	1	1	1	1	1							1					1	1		1	1		1
2		1	1	1	1		1	1						1				1	1			1	1		1
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6		1	1	1	1			1	1					1					1	1	1	1			1
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10		1	1				1	1	1	1				1		1	1	1	1						1
11			1	1			1	1	1	1				1		1	1		1	1					1
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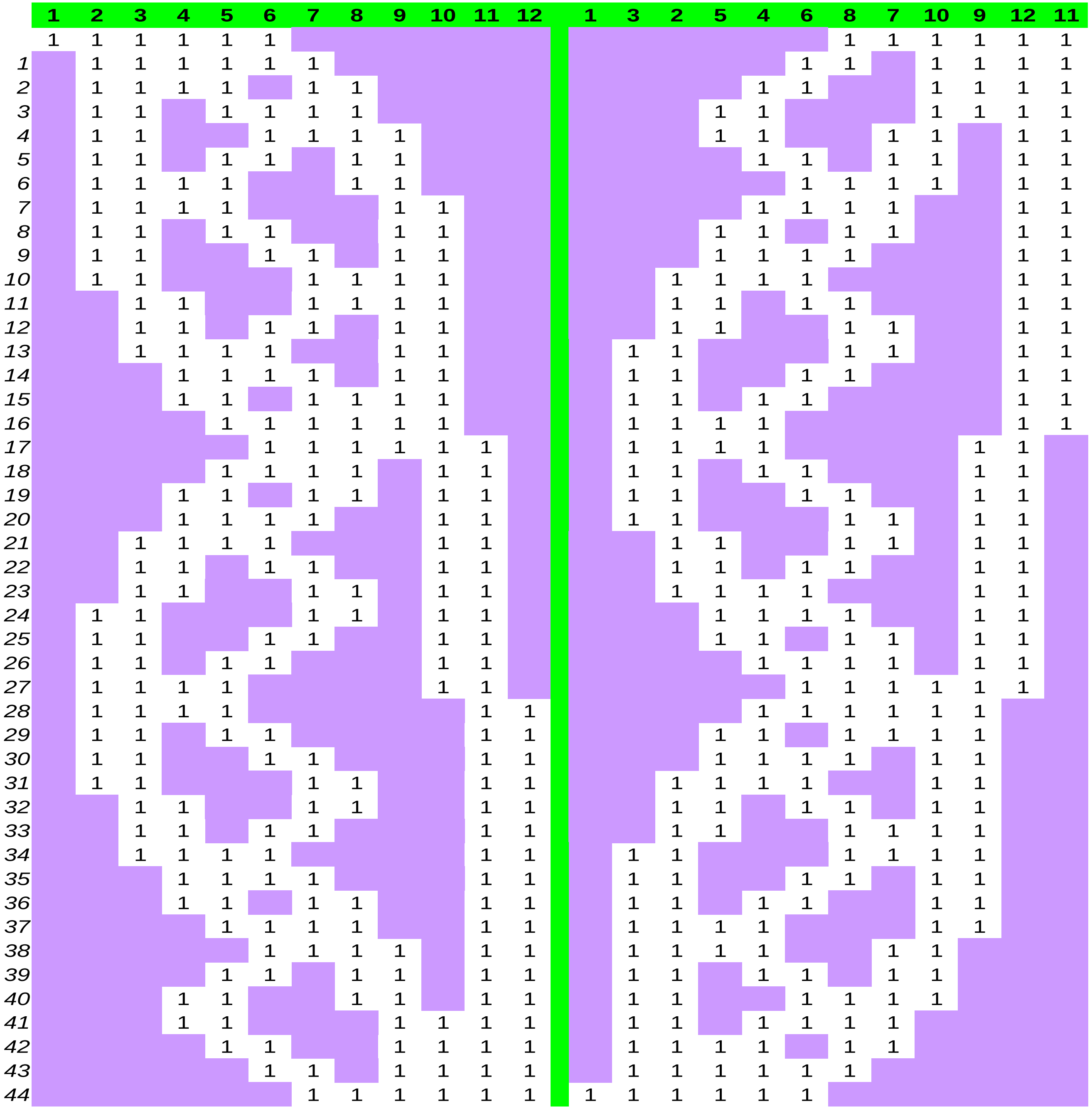
16					1	1	1	1	1	1				1	1	1	1	1							1
17						1	1	1	1	1	1			1	1	1	1	1							
18							1	1	1	1	1	1		1	1	1	1	1	1						

A(6) = path for d=6, label 1



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1		1	1	1	1	1	1											1	1		1	1	1	1
2		1	1	1	1		1	1									1	1			1	1	1	1
3		1	1		1	1	1	1								1	1				1	1	1	1
4		1	1			1	1	1	1							1	1			1	1		1	1
5		1	1		1	1		1	1								1	1		1	1		1	1
6		1	1	1	1			1	1									1	1	1	1		1	1
7		1	1	1	1				1	1							1	1	1	1			1	1
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34			1	1	1	1					1	1		1	1				1	1	1	1		
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38						1	1	1	1		1	1		1	1	1	1			1	1			
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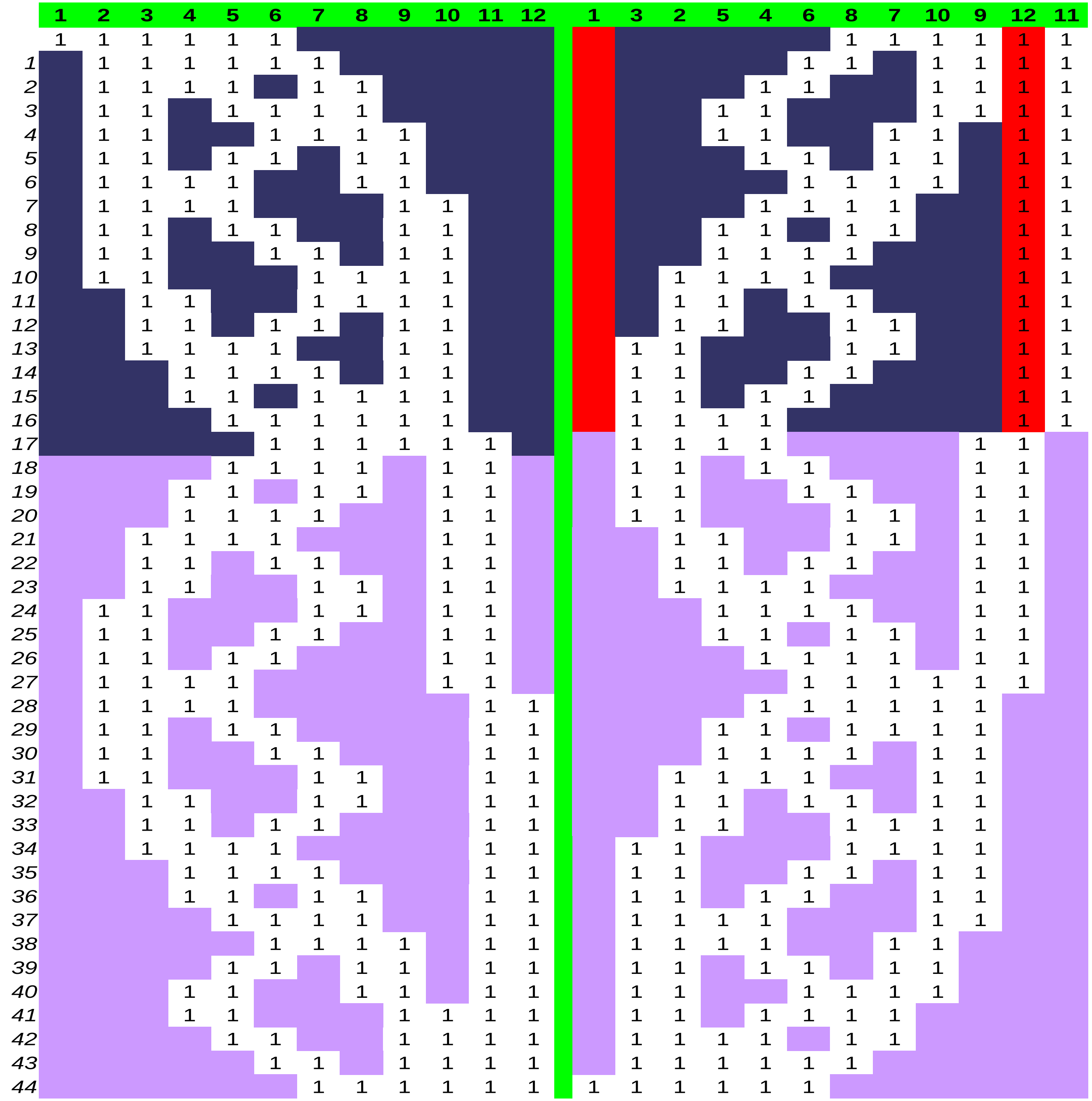
41				1	1				1	1	1	1		1	1		1	1	1	1				
42					1	1			1	1	1	1		1	1	1	1		1	1				
43						1	1		1	1	1	1		1	1	1	1	1	1					
44							1	1	1	1	1	1	1	1	1	1	1	1						

B(6) is prefix of A(6)



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1		1	1	1	1	1	1											1	1		1	1	1	1
2		1	1	1	1		1	1									1	1			1	1	1	1
3		1	1		1	1	1	1								1	1				1	1	1	1
4		1	1			1	1	1	1							1	1			1	1		1	1
5		1	1		1	1		1	1								1	1		1	1		1	1
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43						1	1		1	1	1	1		1	1	1	1	1	1					
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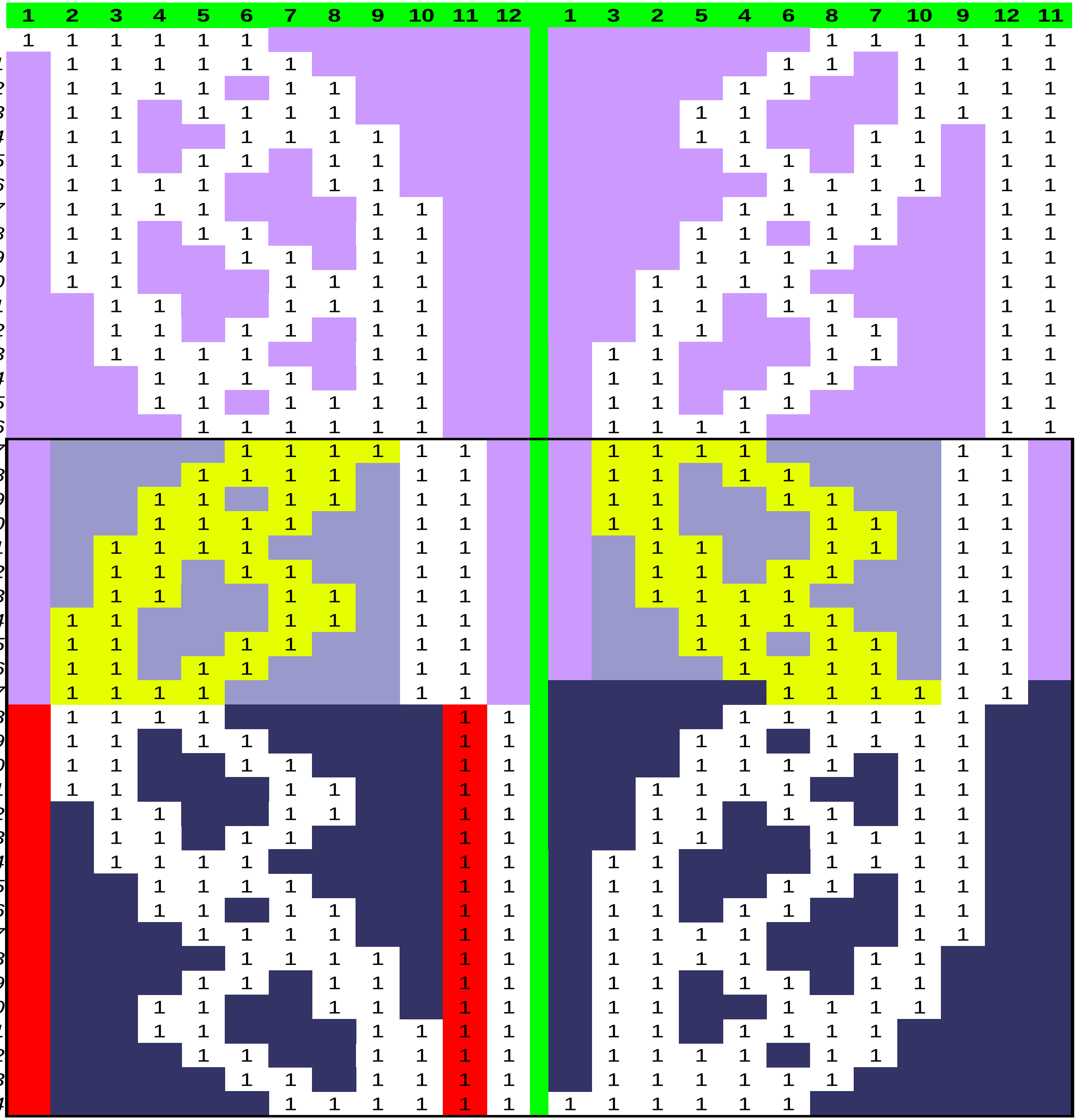
Suffix of A(6) = C(6) = A(4)+B(6)



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3		1	1		1	1	1	1								1	1				1	1	1	1
4		1	1			1	1	1	1							1	1			1	1		1	1
5		1	1		1	1		1	1								1	1		1	1		1	1
6		1	1	1	1			1	1									1	1	1	1		1	1
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9		1	1			1	1		1	1						1	1	1	1				1	1
10		1	1				1	1	1	1					1	1	1	1					1	1
11			1	1			1	1	1	1					1	1		1	1				1	1
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38						1	1	1	1		1	1		1	1	1	1			1	1			
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43						1	1		1	1	1	1		1	1	1	1	1	1					
44							1	1	1	1	1	1	1	1	1	1	1	1						

Recurrences for longest paths

A(d) = LH path dropping label 1 in dim d

B(d) = LH path dropping label 2d in dim d

C(d) = suffix of A(d)

lengths of

B(2) C(2) **A**(2) B(4) C(4) **A**(4) B(6) C(6) **A**(6) . . .

are the **Fibonacci** numbers

2 3 5 8 13 21 34 55 89 . . .

Exponential path lengths

longest paths: drop label 1 or 2d, paths A(d), B(d)

path length $\boxed{\Omega(\phi^{3d/2})}$

with Golden Ratio $\phi = (\sqrt{5} + 1) / 2 = 1.618\ldots$

shortest paths: drop label 3d/2, $\boxed{\text{path } B(d/2)+B(d/2+2)}$

path length $\boxed{\Omega(\phi^{3d/4}) = \Omega(\mathbf{1.434\ldots}^{d})}$

So far:

- NE of a bimatrix game = combinatorial polytope problem
- label dual cyclic polytopes,
  equilibrium at end of **exponentially long** paths
- **but**: fully mixed equilibrium easily guessed
  by support enumeration algorithms

Extension to nonsquare games

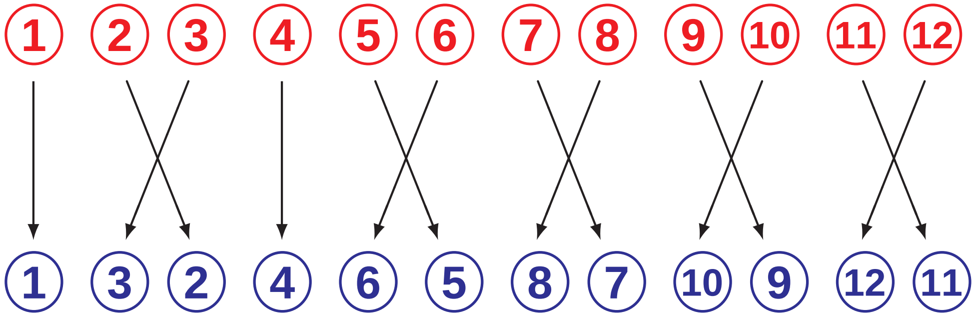
- Extend to $d \times 2d$ games with hard-to-guess support (exponentially many guesses on average) **and** exponentially long paths

- Use dual cyclic polytopes with similar labeling as for $d \times d$ games.

Nonsquare games

P = dual cyclic polytope in dimension **d** with 3d facets

Q = dual cyclic polytope in dimension **2d** with 3d facets



P’s facets labeled ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨ ⑩ ⑪ ⑫

Q’s facets labeled ① ③ ② ④ ⑥ ⑤ ⑧ ⑦ ⑩ ⑨ ⑫ ⑪

equilibria:

0 0 0 0 1 1 1 1 0 0 0 0   1 1 1 1 0 0 0 0 1 1 1 1

0 0 0 0 1 1 0 0 1 1 0 0   1 1 1 1 0 0 1 1 0 0 1 1

0 0 0 0 0 1 1 0 0 1 1 0   1 1 1 1 0 1 1 0 0 1 1 0

0 0 0 0 0 0 1 1 0 1 1 0   1 1 1 1 1 1 0 0 0 1 1 0

. . .

Equilibrium supports = exponentially small fraction of all supports