COMP 360: Algorithm Design Techniques
Tutorial given on March 17, 2004
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Diet Problem (example of a Linear Program, from Vašek Chvátal's Linear Programming)

Food	Serving Size	Energy (Calories)	Protein (grams)	Calcium (milligrams)	Price per serving (cents)
Oatmeal	28 g	110	4	2	3
Chicken	100 g	205	32	12	24
Eggs	2 large	160	13	54	13
Whole milk	237 ml	160	8	285	9
Cherry pie	170 g	420	4	22	20
Pork with beans	260 g	260	14	80	19

Define:

x_1: Daily portions of oatmeal

…

x_6: Daily portions of pork with beans

Add these constraints:

We need at least 2000 calories:

$$110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 \geq 2000$$

We need at least 55 grams of protein:

$$4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 \geq 55$$

We need at least 800 milligrams of calcium:

$$2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 \geq 800$$

Limits on servings per day:

At most 4 servings of oatmeal per day $0 \leq x_1 \leq 4$
At most 3 servings of chicken per day $0 \leq x_2 \leq 3$
At most 2 servings of eggs per day $0 \leq x_3 \leq 2$
At most 8 servings of milk per day $0 \leq x_4 \leq 8$
At most 2 servings of cherry pie per day $0 \leq x_5 \leq 2$
At most 2 servings of pork with beans per day $0 \leq x_6 \leq 2$

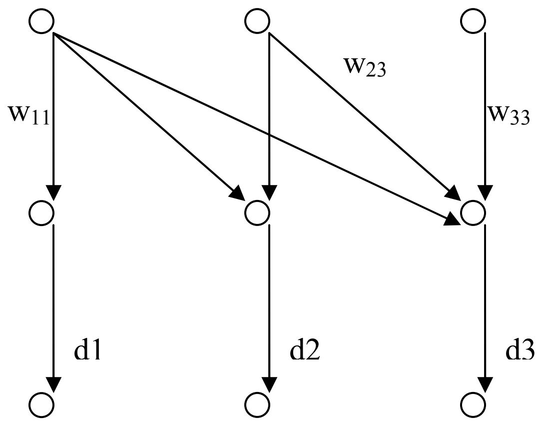
We want to minimize the total daily cost of the diet.

⇒ **Minimize this objective function:**

$$3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6$$

Lot sizing (example of an Integer Linear Program, from David Avis)

We own a factory that manufactures some product. We have to decide on a production schedule over the next 3 periods (say periods are weeks). We have specific demands in each period. We can produce and store units in prevision of helping meet demand in one of the later periods, but this will add storage costs to our total costs. Our objective is to minimize total cost.



Define:

d_i: demand in week i

w_{ij}: production at week i used to supply part of d_j ($i \leq j$)

Then we need constraints to make sure that demands are exactly met in each week:

Demand is met in the first week: $\mathbf{w_{11} = d_1}$

Demand is met in the second week: $\mathbf{w_{12} + w_{22} = d_2}$

Demand is met in the third week: $\mathbf{w_{13} + w_{23} + w_{33} = d_3}$

We also need the obvious constraints that state that all variables representing produced quantities need to be non negative:

$\mathbf{w_{ij} \geq 0}$ (for $1 \leq i \leq j \leq 3$)

Define:

p_i: cost of producing one unit in week i

h_i: cost of keeping one unit in storage during week i

Then we can define c_{ij}, the cost of producing a unit in week i and storing it until week j:

$c_{ij} = p_i + h_i + h_{i+1} + \ldots + h_j$

We want to **minimize** the following objective function, which represents total cost:

$$\sum_{i=1}^{3} \sum_{j=i}^{n} c_{ij} w_{ij}$$

Or more explicitly:

$$\mathbf{c_{11}w_{11} + c_{12}w_{12} + c_{13}w_{13} + c_{22}w_{22} + c_{23}w_{23} + c_{33}w_{33}}$$

To be more realistic, we could also decide that producing in a certain week i implies a fixed cost f_i. This fixed cost is not charged if no units at all are produced in week i. How do we incorporate this into our problem?

We need more variables:

$y_i = 1$ if we produce in week i, or 0 if we don't

Then it's easy to incorporate the fixed costs into our objective function, which becomes

$$c_{11}w_{11} + c_{12}w_{12} + c_{13}w_{13} + c_{22}w_{22} + c_{23}w_{23} + c_{33}w_{33} + \mathbf{y_1f_1 + y_2f_2 + y_3f_3}$$

To summarize, here is the complete formulation of this integer programming problem:

Minimize

$$\mathbf{c_{11}w_{11} + c_{12}w_{12} + c_{13}w_{13} + c_{22}w_{22} + c_{23}w_{23} + c_{33}w_{33} + y_1f_1 + y_2f_2 + y_3f_3}$$

Subject to :

$$\mathbf{w_{11} = d_1}$$
$$\mathbf{w_{12} + w_2 = d_2}$$
$$\mathbf{w_{13} + w_{23} + w_{33} = d_3}$$
$$\mathbf{w_{11}, w_{12}, w_{13}, w_{22}, w_{23}, w_{33} \geq 0}\textbf{, and integer}$$
$$\mathbf{y_1, y_2, y_3 \geq 0, \leq 1}\textbf{, and integer}$$

where the c's and f's are either known quantities or can be precomputed from known quantities.

<u>Quiz from section 1.3 of Komei Fukuda's notes</u>

1. Max $2x + 4y$
   Subject to
   $x - 3y = 5$
   $y \leq 0$

   This is a valid LP.

2. Max $2x + 4y$
   Subject to
   $x - 3y = 5$
   $x \geq 0$ or $y \leq 0$

   This is not a valid LP, because there is no way to represent this kind of OR with linear constraints.

3. Max $x + y + z$
   Subject to
   $x + xyz \leq 5$

$x - 5y \geq 3$

This is not a valid LP, because the first constraint is not linear.

4. Min $x^2 + 4y^2 + 4xy$
   Subject to
   $x + 2y \leq 4$
   $x - 5y \geq 3$
   $x \geq 0,\ y \geq 0$

   This is not a valid LP, because the objective function is not linear.

5. Min $x_1 + 2x_2 - x_3$
   Subject to
   $x_1 \geq 0,\ x_2 \geq 0$
   $x_1 + 4x_2 \leq 4$
   $x_2 + x_3 \leq 4$
   x_1, x_2, x_3 are integers

   This is a valid LP. In particular, it's an ILP (Integer Linear Program) because all its variables are required to be integer.

6. Min $2x_1x_2 - x_3$
   Subject to
   $x_1 + 4x_2 \leq 4$
   $x_2 + x_3 \leq 4$
   $x_1 \geq 0,\ x_2 \geq 0$
   x_1 is integer

   This is not a valid LP, because the objective function is not linear.

7. Min $x_1 + 2x_2 - x_3$
   Subject to
   $x_1 \geq 0,\ x_2 \geq 0$
   $x_1 + 4x_2 \leq 4$
   $x_2 + x_3 \leq 4$
   x_1, x_2, x_3 are either 0 or 1.

   This is a valid LP. The difference between this OR and the one in exercise 1 is that this one can be represented as follows:
   $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$
   $x_1 \leq 1,\ x_2 \leq 1,\ x_3 \leq 1$
   x_1, x_2, x_3 are integers

   We see that this is actually an ILP because all its variables are required to be integer.

Proving unboundedness using a certificate

Primal problem: Max $x_1 + x_2$

Subject to

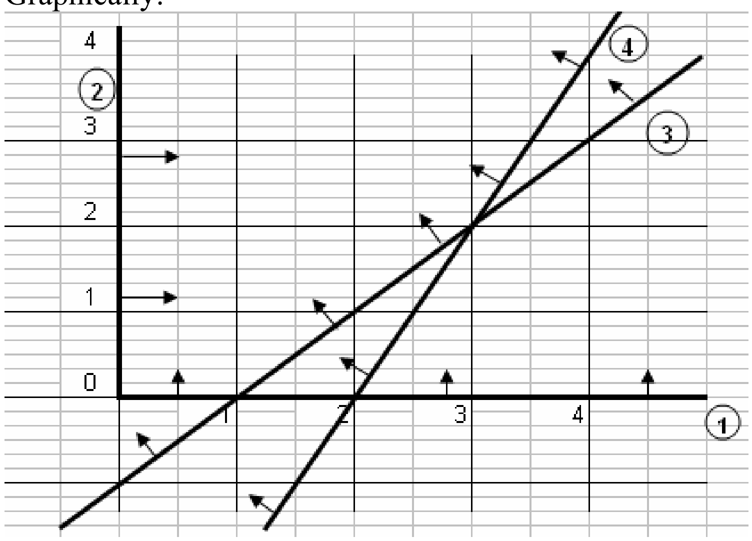
(1) $x_1 \geq 0$

(2) $x_2 \geq 0$

(3) $x_1 - x_2 \leq 1$

(4) $2x_1 - x_2 \leq 4$

Graphically:



Dual problem: Min $y_1 + 4y_2$

Subject to

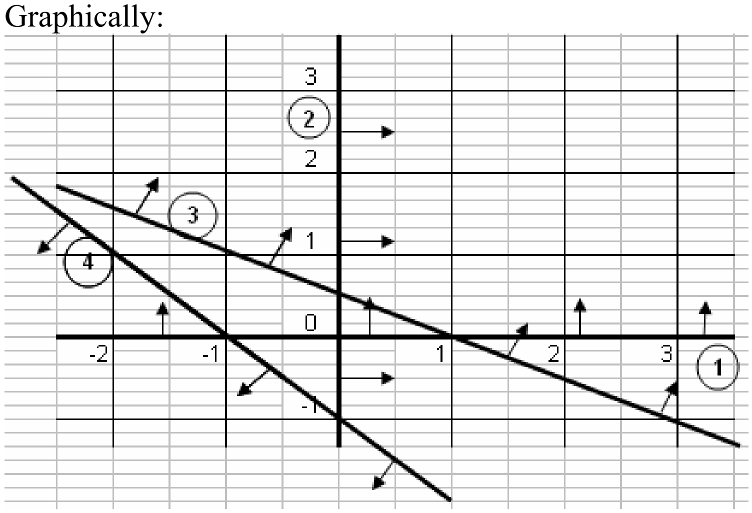
(1) $y_1 \geq 0$

(2) $y_2 \geq 0$

(3) $y_1 + 2y_2 \geq 1$

(4) $-y_1 - y_2 \geq 1$

Graphically:



Graphically it becomes obvious that the primal is unbounded, and that the dual is infeasible. We prove this more formally using a certificate. Here is Theorem 2.5 from Komei Fukuda's notes:

Max $c^T x$ subject to $Ax \le b$ and $x \ge 0$
is unbounded iff

1) it has a feasible solution x
2) there exists a direction z such that $z \ge 0$, $Az \le 0$, and $c^T z > 0$

Let x = (0, 0): this is a valid feasible solution to our primal problem, so 1 is satisfied.

Now to verify 2, we need to find some z = (z1, z2) such that

$z1 \ge 0,\ z2 \ge 0$

$$\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} z1 \\ z2 \end{bmatrix} \le 0$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} z1 \\ z2 \end{bmatrix} > 0$$

Check that z = (1, 2) satisfies the above. Therefore our pair (x, z) is a certificate for the unboundedness of the primal problem.

<u>A note about unconstrained variables in lp_solve</u>

lp_solve requires all variables to be non negative. So how do we model a variable x that may assume negative values?

We introduce two more variables y and z, and everywhere x appears in our objective function or in our constraints, we replace x by (y-z).