COMP 360: Algorithm Design Techniques
Tutorial given on April 7, 2004
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Poly-reducibility Exercise

Assume problem P1 is reducible in polynomial time to problem P2.

Claim: If both P1 and P2 are NP-Complete, then P2 is poly-reducible to P1.

Proof:
Since SAT is NP-Complete and P2 is in NP, there exists a polynomial time transformation T from P2 to SAT. Also, since P1 is NP-Complete, there is a sequence T1, …, Tn of polynomial time transformations that starts from an instance of SAT and eventually yields an instance of P1. Then we can use the sequence of transformations T, T1, …, Tn to reduce P2 to P1 in polynomial time.

Relations between Clique, Vertex Cover, and Independent Set

Given: A graph G = (V, E), and V' $\subseteq$ V.
The following statements are equivalent:

a) V' is a vertex cover for G
b) V – V' is an independent set for G
c) V – V' is a clique in $\overline{G}$, the complement of G

Proof of a) $\Rightarrow$ b) by contradiction

Assume V' is a VC for G. Suppose V-V' is not an independent set for G. Then there is some edge e=(u,v) in G such that both u and v are in V-V'. This implies that neither u nor v are in V', therefore edge e is not covered by V', and hence V' is not a VC for G.

Proof of b) $\Rightarrow$ a) by contradiction

Assume V – V' is an independent set for G. Suppose V' is not a VC for G. Then there is some edge e = (u,v) such that neither u nor v is in V'. Then both u and v are in V – V'. But then since u and v share an edge, V – V' can't be an independent set for G.

Proof of a) $\Rightarrow$ c) by contradiction

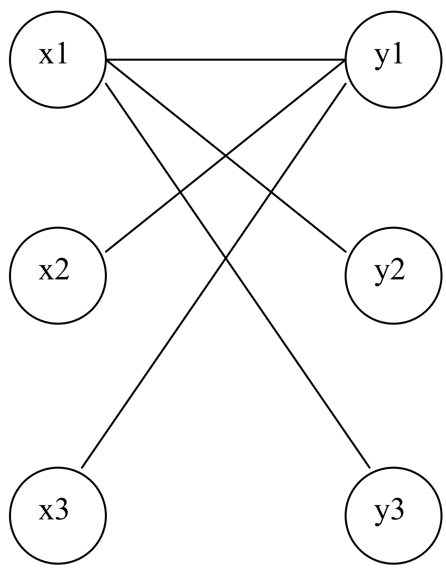
Assume V' is a VC for G. Suppose V-V' is not a clique in $\overline{G}$. Then there exists some pair of vertices u, v in V-V' that don't share an edge in $\overline{G}$. But then u and v do share an edge in the original graph G. Then V' can't be a VC for G because u and v are not in V' yet they share an edge.

Proof of c) ⇒ a) by contradiction

Assume V – V' is a clique in $\overline{G}$. Suppose V' is not a VC in G. Then there exists an edge e = (u, v) of G such that neither u nor v are in V'. Since that edge is in G, it's not in $\overline{G}$, and therefore u and v can't be part of a clique in $\overline{G}$. Therefore V – V' is not a clique in $\overline{G}$.

<u>Bipartite graphs</u>

Say we're interested in finding the largest independent set in a bipartite graph. Intuition may lead us to believe that it can be found by choosing the largest of the two partite sets X, Y. But consider this counterexample:



Here the largest independent set is neither X nor Y, but rather the set {x2, x3, y2, y3}.

So we need another strategy to find the largest independent set in a bipartite graph. We already know of an algorithm to find a max matching in a bipartite graph using network flows. Keeping this in mind, the relationship outlined above between independent set and vertex cover should suggest an algorithm to find the largest independent set in a bipartite graph.