Review

- Recursive formula gives recursive algorithm - but algorithm often takes exponential time.
- 'Memoization' used to speed up algorithms.
- Store values already computed. At each call we check to see if value already computed.
- Optimal solution to problem gives optimal solution to subproblems.
- Dynamic programming used to optimize.
- DAGs and topological sorts.

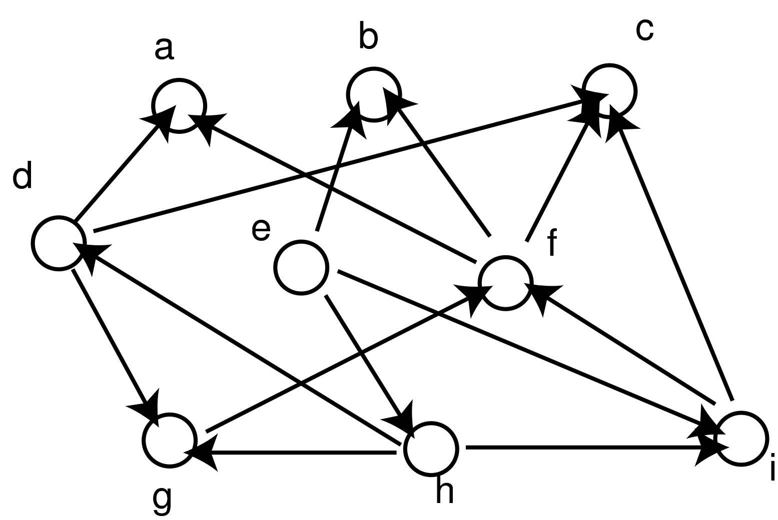
Topological sort

A **topological sort** of a digraph is an ordering $v_1, v_2, \ldots, v_n$ of the vertex such that if (v_i, v_j) is an edge then $i < j$.

We can use induction and the previous lemma to prove:

Lemma
Every DAG has a topological sort

Example 2.1:



Topological sort: $e, h, d, g, f, i, b, a, c$. There are others.

Representing directed graphs

We use an **adjacency list** representation. For each vertex u we list the vertices v such that $(u, v) \in E$.

e.g. for Example 2.1:

v	$Adj[v]$	$Pred[v]$
a		
b		
c		
d	a, g	
e	b, h, i	
f	a, b, c	
g	f	
h	d, g, i	
i	c, f	

From this, we can easily compute a **predecessor table**. For each v, find the list of vertices u such that $(u, v) \in E$.

$FindPreds(G)$
1. Initialise $Prev[v] \leftarrow \emptyset$ for all $v \in V$.
2. **for** all $u \in V$ **do**
3.   **for** all $v \in Adj[u]$ **do**
4.   Add u to the end of the list $Pred[v]$.

Project scheduling

We wish to complete a project/computation in the shortest amount of time possible. We can perform tasks in parallel, but there are some tasks that we have to finish before we can begin others.

Input:

- Set of tasks $t_1, \ldots, t_n$.
- Set P of constraints (t_i, t_j). The pair (t_i, t_j) means that task t_i must be completed before task t_j can begin.
- Task t_i takes $f(t_i)$ time to run.

Question: If we are allowed unlimited parallel processors, how long does it take to complete all of the tasks?

Example

task	time	must be done after:
A	3	B
B	3	
C	5	

We start with B and C and run A after B has finished. Total time is 6.

Solution - recursion

Let $g(t_i)$ denote the first possible time that t_i could finish.

• If there are no precedent constraints on t_i then we can run t_i straight away, and $g(t_i) = f(t_i)$.

• If there are precedent constraints, then we have to wait until they have all finished before we can run t_i. Hence

$$g(t_i) = \max\{g(t_j) : (t_j, t_i) \in P\} + f(t_i)$$

1. Construct G with vertices $\{t_1, \ldots, t_n\}$ and edge set $E = P$.
2. Construct the $Pred$ tables for G.
3. Initialise $g[t_i] \leftarrow -1$ for all i.
4. Call $FinishTime(t_i)$ for each t_i.
5. Return the maximum of $g[t_i]$ for $i = 1, 2, \ldots, n$.

$FinishTime(t_i)$

1. **If** $g[t_i] \geq 0$ **then**
2.   **return** $g[t_i]$
3. **else**
4.   **if** $Pred[t_i]$ is empty **then**
5. $g[t_i] \leftarrow f[t_i]$
6.   **else**
7. $g[t_i] \leftarrow \max\left\{FinishTime(t_j) : t_j \in Pred[t_i]\right\} + f(t_i)$
8.   **return** $g[t_i]$

Longest path

How long is the longest path in a DAG? This is a hard problem in a regular digraph, but can be easily solved in a DAG.

Similar to project scheduling. Let $l(v)$ denote the longest path ending in vertex v. Then

- $l(v) = 0$ if v is a source.
- Otherwise $l(v) = \max\{l(u) : u \in Pred[v]\} + 1$

1. Initialise $L[v] \leftarrow -1$ for all v.
2. Construct the $Pred$ tables for G.
4. Call $LongPath(v)$ for each v.
5. Return the maximum of $L[v]$ for $i = 1, 2, \ldots, n$.

$LongPath(v)$

If $L[v] \geq 0$ **then**
  return $L[v]$
else
  if $Pred[v]$ is empty **then**
    $L[v] \leftarrow 0$
  else
    $L[v] \leftarrow 1 + \max\{LongPath(u) : u \in Pred[v]\}$
  return $L[v]$.

Features of Dynamic Programming

- Problem divided into *overlapping* sub-problems. (Compare divide and conquer, where subproblems are independent)

- The solution of a subproblem generally depends on solutions of further subproblems.

- A straight recursive solution leads to an exponential time algorithm.

- Solutions of subproblems are stored, as they are often used multiple times during computation.

We can denote dependencies between subproblems using a directed graph. Each subproblem corresponds to one vertex. An edge from subproblem u to subproblem v if we use the solution of u in the solution of v.

Dynamic programming works if and only if this directed graph is acyclic.

General problems in Dynamic Programming

1) How to avoid recursion? (function calls generally take a lot of time).

2) How to extract an optimal solution? (e.g. highest scoring path, longest path).

3) How to handle multiple optimals? (e.g. counting the number of optimal solutions).

Avoiding recursion

Suppose that vertices in G are labelled $v_1, v_2, \ldots, v_n$. A "simpler" alternative to $LongPath$ might be:

$WrongLongPath(G)$
1. **for** $i \leftarrow 1$ to n **do**
2. **if** v_i is a source **then**
3. $L[v_i] \leftarrow 0$
4. **else**
5. $L[v_i] \leftarrow 1 + \max\{L[v_j] : v_j \in Pred[v_i]\}$.

Problem: In step 5 we have no guarantee that the values $L[v_j]$ have already been initialised and computed.

Solution: First construct a topological sort $v_1, v_2, \ldots, v_n$ of the vertices. Then $v_j \in Pred[v_i]$ will imply that $(v_j, v_i) \in E$, so v_j comes before v_i in the ordering.

Principle:

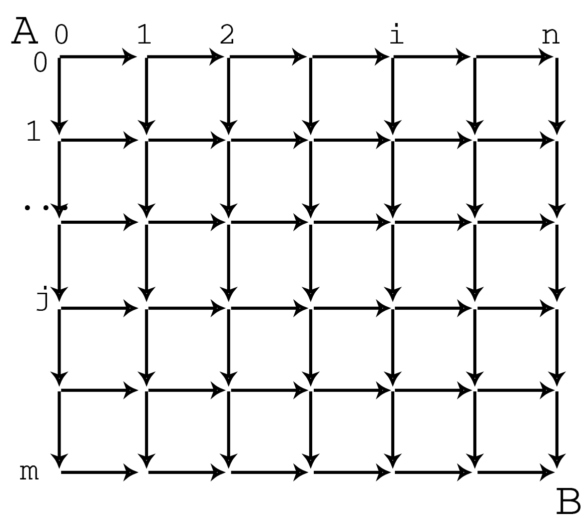
If you are using loops with Dynamic Programming, make sure that subproblems are solved in a topological order, with respect to the dependency graph.

Example: flag collecting

Subproblems: one for each (i, j). The subproblem is "What is the maximum score of a path from $(0, 0)$ to (i, j) that always goes downwards or to the right"

To solve the problem for (i, j) we need to have solved $(i - 1, j)$ and $(i, j - 1)$.

The dependency digraph is a directed version of the grid, with edges directed downwards and rightways.



One "topological sort" of this graph is given by the loops

for $i \leftarrow 0$ to n **do**
  for $j \leftarrow 0$ to m **do**
    Compute $G[i, j]$

Recursion free algorithm for flag collecting

GoodPaths2

for $i \leftarrow 0$ to n **do**
  for $j \leftarrow 0$ to m **do**
    if $i = 0$ and $j = 0$ **then**
      $G[i,j] \leftarrow 0$
    else if $i > 0$ and $j = 0$ **then**
      $G[i,j] \leftarrow G[i-1,0] + H[i,0]$
    else if $i = 0$ and $j > 0$ **then**
      $G[i,j] \leftarrow G[0,j-1] + V[i,0]$
    else
      $G[i,j] \leftarrow \max\{G[i-1,j] + H[i,j], G[i,j-1] + V[i,j]\}$
return $G[n,m]$

Note: whenever we look-up $G[x,y]$ we know its already been computed.