LINEAR PROGRAMMING BRINGS MARITAL BLISS

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A stable matching is an assignment of n men to n women so that no two people prefer each other to their respective spouses. This paper describes the convex hull of the incidence vectors of stable matchings. With this description, one may solve the optimal stable marriage problem as a linear program.

stable matching * linear programming

1. Introduction

The stable marriage problem asks whether there is a matching of n men to n women so that no two people prefer each other to their respective spouses; that is, does there exist a *stable matching*? Gale and Shapley [6] resolved this question by showing that regardless of the individual preferences, a stable matching always exists. In fact, they described a procedure for finding the stable matching that gives each man his best possible mate; that is, in no stable matching could any man be paired with someone he finds more desirable.

McVitie and Wilson [12] pointed out that although the men all agree Gale and Shapley's stable matching is best, the women all agree it is worst; that is, in no stable matching could any woman be paired with a man she finds less desirable. Thus, investigators turned to the problem of finding equitable or socially optimal stable matchings.

In the optimal stable marriage problem, each possible marriage has a social value and we are asked to find a stable matching of maximum total value. Irving et al. [9] provided an ingenious, albeit complex, algorithm for finding an optimal

stable matching by exploiting the one-to-one correspondence between stable matchings and the closed subsets of a certain partially ordered set.

In this paper, we show how to find an optimal stable matching by more conventional means: linear programming. We formulate the optimal stable marriage problem as a small linear program and show that its extreme points are all integer valued. Thus, solving the linear program solves the optimal stable marriage problem.

Dantzig [5] referred to the linear programming characterization of the bipartite matching problem as 'a mathematical proof that of all the possible forms of marriage (monogamy, bigamy, polygamy, etc.) monogamy is the best'. Our linear programming characterization of the stable marriage problem extends Dantzig's observation by showing that among all forms of stable marriage, monogamous stable marriage is the best.

2. An integer programming formulation

We write $x >_z y$ to denote that person z prefers person x to person y. Thus, a matching μ (which assigns the woman $\mu(m)$ to man m and the man $\mu(w)$ to woman w) is stable if and only if there is no pair m and w such that both $w >_m \mu(m)$ and $m >_w \mu(w)$.

For the sake of presentation, we assume that the preferences of each individual form a complete

order over all the members of the opposite sex and that all possible marriages are acceptable, that is, it is better to marry than not. Once we describe a linear programming formulation of the stable marriage problem under these restrictions, it is an easy exercise to eliminate the assumption that all possible mates are acceptable. It is not clear how to extend these results to the situation in which an individual may express indifference between two possible mates.

Let M and W be the sets of men and women, respectively. The incidence vector of a matching μ is $x \in \{0, 1\}^{|M| \times |W|}$ such that $x(m, w) = 1$ if $\mu(m) = w$ and $x(m, w) = 0$, otherwise. We write

$x(m >, w)$ for $\Sigma(x(i, w)$: for all men i
           such that $m >_w i)$,

$x(m' > i > m, w)$ for $\Sigma(x(i, w)$: for all men i
           such that $m' >_w i >_w m)$,

$x(M, w)$ for $\Sigma(x(i, w)$: $i \in M)$.

Thus, if x is the incidence vector of a matching μ, then $x(m >, w) = 1$ when woman w is married to someone she finds less desirable than man m, that is, $m >_w \mu(w)$. Similarly, $x(m' > i > m, w) = 1$ when woman w is married to someone she finds less desirable than man m', but more desirable than man m, that is, $m' >_w \mu(w) >_w m$. Finally, $x(M, w) = 1$ since woman w is married under μ.

Similarly, we write

$x(m, > w)$ for $\Sigma(x(m, i)$: for all men i
           such that $i >_m w)$,

$x(m, w >)$ for $\Sigma(x(m, i)$: for all men i
           such that $w >_m i)$,

$x(m, W)$ for $\Sigma(x(m, i)$: $i \in W)$.

With this notation, we can characterize the incidence vector of a stable matching as an integer vector x satisfying

$x(m, W) = 1$

     for each $m \in M$,   (1)

$x(M, w) = 1$

     for each $w \in W$,   (2)

$x(m >, w) - x(m, > w) \leqslant 0$

     for each $(m, w) \in M \times W$,   (3)

$x(m, w) \geqslant 0$

     for each $(m, w) \in M \times W$.   (4)

Constraints (1) and (2) require that each man marry one woman and each woman marry one

man. In asserting that among all forms of marriage monogamy is best, Dantzig was referring to the fact that each extreme point of the linear program described by constraints (1), (2) and (4) is the incidence vector of a matching [5].

Constraints (3) impose the stability condition: If woman w marries someone less desirable than m, then man m must marry someone more desirable than she. Note that together with constraints (1), (3) also implies that if man m marries someone less desirable than woman w, she must marry someone more desirable than he. Thus, an integer vector $x \in \mathbb{R}^{|M| \times |W|}$ is the incidence vector of a stable matching if and only if x satisfies (1)–(4).

In general, finding an integer solution to a set of linear inequalities is computationally very difficult [7]. We show, however, that we may ignore the integrality condition and consider only the linear inequality system (1)–(4) in finding an optimal stable matching. In particular, we show that each extreme point of this linear system is integer valued. Note that the constraints (1), (2) and (4) describe the well-known (perfect) bipartite matching polytope all of whose extreme points are integer valued [5]. Further, constraints (3) and (4) together describe a cone. We show that the intersection of these two polyhedra gives rise to a polytope with integer extreme points.

3. A linear programming formulation

In this section we show that (1)–(4) is a linear inequality description of the stable marriage problem; that is, we show that each extreme point solution to (1)–(4) is integer valued. The first ingredient in our argument is Gale and Shapley's algorithm for finding the 'male-optimal' stable matching. In this algorithm, the men propose to the women who either consider or reject them as described below.

Algorithm 1. The deferred acceptance algorithm. In a round, each man proposes to his favorite woman from among those who have not yet rejected him. Each woman rejects all but her best proposal. Rounds continue until each woman holds a proposal.

At the end of Algorithm 1, the men and women are tentatively paired according to the male-opti-

mal stable matching μ_M. Now, reverse the roles of the men and women in Algorithm 1. This procedure, henceforth referred to as Algorithm 2, terminates with the men and women paired according to the female-optimal stable matching μ_F [6].

The sequence of proposals and rejections in the Deferred Acceptance Algorithm identifies many variables which must be zero in every solution to (1)–(4).

Lemma 1. *For each $x \in \mathbb{R}^{|M| \times |W|}$ satisfying (1)–(4), if woman w received a proposal from man m in* Algorithm 1, *then $x(m >, w) = 0$. Moreover, if she rejected his proposal, then $x(m, w) = 0$. Similarly, if man m received a proposal from woman w in* Algorithm 2, *then $x(m, w >) = 0$ and, if he rejected her proposal, then $x(m, w) = 0$.*

Proof. Consider a vector x satisfying (1)–(4). Suppose woman w received a proposal from man m in the first round of Algorithm 1. Then she is his favorite and so $x(m, > w) = 0$, trivially. Since x satisfies constraint (3) for the pair (m, w), $x(m >, w) = 0$. Moreover, if she rejected his offer in the first round, she must have received a proposal from some man m' whom she prefers. Thus, $x(m' >, w) = 0$ and, in particular, $x(m, w) = 0$.

Similar arguments show that if man m received a proposal from woman w in the first round of Algorithm 2, then $x(m, w >) = 0$ and, if he rejected her, then $x(m, w) = 0$.

Finally, proceeding by induction on the round in which the proposal was received or rejected proves the lemma. □

In light of Lemma 1, we remove man m and woman w from each other's preference lists if at the end of Algorithm 1, she holds a proposal from someone she prefers or, at the end of Algorithm 2, he holds a proposal from someone he prefers. We refer to the resulting preference lists as the *short lists*. (Note that our definition differs slightly from that of Irving et al. [9], as we have also removed those pairs eliminated in Algorithm 2.) One immediate consequence of Lemma 1 is that women w is first on man m's short list if and only if he is last on hers. Likewise, man m is first on woman w's short list if and only if she is last on his.

Although the male-optimal stable matching μ_M obtained by assigning each man to the first wo-

man on his short list is the best stable matching from the men's point of view, it is the worst from the women's point of view. Thus, in order to obtain better partners, some of the women reject the proposals they hold at the end of Algorithm 1.

Suppose for example, that woman w_1 rejects her proposal from man m_1. Then the best he can do is to propose to the next woman on his short list, say woman w_2. Since woman w_2 prefers this new proposal over the one she currently holds, say from man m_2, she will in turn reject man m_2. Again, the best m_2 can do is to propose to the next woman on his short list, say woman w_3.

Thus, woman w_1's ambitions precipitate a sequence of rejections and proposals, which continues until either some man is rejected and has no one left to propose to, or a cycle is formed when some man proposes to a woman who precedes him in the sequence. This latter case gives rise to a *rotation* or cycle $\rho = \{(m_j, w_j) : j \in [1, \dots, r]\}$ such that:

    (i) For each $j \in [1, \dots, r]$, woman w_j is first on man m_j's short list.

    (ii) For each $j \in [1, \dots, r-1]$, woman w_{j+1} is second on man m_j's short list, and to complete the cycle.

    (iii) Woman w_1 is second on man m_r's short list.

Irving et al. introduced the notion of a rotation and showed that if each woman w_j rejects her proposal from man m_j and accepts her proposal from man m_{j-1} (woman w_1 accepts her proposal from man m_r), the resulting matching is stable.

Lemma 2 interprets the sequence of rejections and proposals indicated by a rotation as row operations on (1)–(4). These row operations identify additional variables that must be zero in every solution to (1)–(4) and reformulate the constraints among the pairs in a rotation as network flow constraints.

Lemma 2. *Consider a rotation $\rho = \{(m_j, w_j) : j \in [1, \dots, r]\}$ with respect to the short lists. For each x satisfying (1)–(4),*

    (i) $x(m_j, w_j) = x(m_1, w_1)$ *for each $j \in [2, \dots, r]$,*

    (ii) $x(m_{j-1} > i > m_j, w_j) = 0$, *for each $j \in [2, \dots, r]$,*

    (iii) $x(m_r > i > m_1, w_1) = 0$.

Proof. By Lemma 1,

$$x(m_r, > w_1) = x(m_r, w_r), \tag{5}$$

and

$$x(m_r >, w_1) = x(m_1, w_1) + x(m_r > i > m_1, w_1). \tag{6}$$

Thus, constraint (3) for the pair (m_r, w_1) is equivalent to

$$x(m_1, w_1) + x(m_r > i > m_1, w_1) - x(m_r, w_r) \leqslant 0. \tag{7}$$

Likewise, constraint (3) for each pair (m_{j-1}, w_j), $j \in [2, \ldots, r]$, is equivalent to

$$x(m_j, w_j) + x(m_{j-1} > i > m_j, w_j)$$
$$- x(m_{j-1}, w_{j-1}) \leqslant 0. \tag{8}$$

Combining (7) and (8) we see that $x(m_r > i > m_1, w_1) = 0$, and for each $j \in [2, \ldots, r]$, $x(m_{j-1} > i > m_j, w_j) = 0$, and $x(m_j, w_j) = x(m_1, w_1)$, as desired. $\square$

If $\rho = \{(m_j, w_j): j \in [1, \ldots, r]\}$ is a rotation with respect to the short lists and woman w_1 rejects her proposal from man m_1, then, in light of Lemma 2, we may remove each man m (including man m_j) and woman w_j from each other's short lists if she likes him less than her new mate m_{j-1}. We refer to this process as *eliminating* the rotation ρ and we refer to the resulting preference lists as *reduced lists*.

As in the short lists, woman w is first on man m's reduced list if and only if he is last on hers. Likewise, man m is first on woman w's reduced list if and only if she is last on his. Thus, we may define a rotation $\rho = \{(m_j, w_j): j \in [1, \ldots, r]\}$ with respect to reduced lists in the natural way. If woman w_1 rejects her proposal from man m_1, eliminating the rotation ρ leads to a new set of reduced lists and new rotations.

We must eliminate certain rotations in order to obtain reduced lists in which we can identify a given rotation. For example,

(a) If the pair (m, w) is eliminated by a rotation π and (m, w'), where $w >_m w'$, is in the rotation ρ, we must eliminate π before we can identify ρ.
Similarly:

(b) If the pair (m', w), where $m' >_w m$, is in the rotation ρ, we must eliminate π before we can identify ρ.

Let R be the collection of rotations together with the set $\rho(\mu_F) = \{(m, \mu_F(m)): m \in M\}$. With each pair $(m, w) \in M \times W$ in some member of R, we associate the member of R, denoted $\rho(m, w)$, containing (m, w). We define the M-predecessor of (m, w), denoted $\rho_M(m, w)$, to be the member of R, (if one exists) such that:

(i) $(m, i) \in \rho_M(m, w)$ for some woman i with $i >_m w$; and

(ii) for each j such that $i >_m j >_m w$, (m, j) is not in any member of R.

Finally, we define the W-predecessor of (m, w), denoted $\rho_W(m, w)$, to be the member of R, (if one exists) such that:

(i) $(i, w) \in \rho_W(m, w)$ for some man i with $m >_w i$; and

(ii) for each j such that $m >_w j >_w i$, (j, w) is not in any member of R.

Note that if the pair (m, w) is in some member of R, then $\rho_M(m, w)$ and $\rho_W(m, w)$ are identical. If (m, w) is not in a member of R, this is not necessarily true. Further, if (m, w) is in some member of R, then (a) implies we must eliminate $\rho_M(m, w)$ before eliminating $\rho(m, w)$. If the pair is not in any member of R, then (b) implies we must eliminate $\rho_W(m, w)$ before eliminating $\rho_M(m, w)$.

Before proving that (1)–(4) is a linear inequality description of the stable marriage problem, we state the following existence theory for rotations originally proved in Irving et al.

Lemma 3. *If some man has more than one woman on his reduced lists, then there is a rotation with respect to the reduced lists.*

We are now prepared to show that the linear system (1)–(4) has integer extreme points. In proving this, we interpret eliminating rotations as row operations transforming (1)–(4) to an equivalent network flow problem.

Theorem 1. *For any vector $c = (c_{mw}: m \in M$ and $w \in W)$ of social values, an extreme point optimal solution to the linear program*

max cx

s.t.

$$x(m, W) = 1$$

for each $m \in M$,

$x(M, w) = 1$

for each $w \in W$,

$x(m >, w) - x(m, > w) \leqslant 0$

for each $(m, w) \in M \times W$,

$x(m, w) \geqslant 0$

for each $(m, w) \in M \times W$,

solves the optimal stable marriage problem.

Proof. See Appendix. □

Although the polytope described by (1)–(4) has integer-valued extreme points, its constraint matrix is not generally totally unimodular. To see this, observe that if it were, then the constraint matrix of the following system would be totally unimodular:

$x(m, W) = 1$

for each man $m \in M$,   (9)

$x(M, w) = 1$

for each woman $w \in W$,   (10)

$x(m >, w) + x(m, w) + x(m, w >) \leqslant 1$

for each $(m, w) \in M \times W$,   (11)

$x(m, w) \geqslant 0$

for each $(m, w) \in M \times W$.   (12)

But, the submatrix consisting of the columns for $x(m, w)$, $x(m', w)$ and $x(m, w)$ and the rows (9) for man m, (10) for woman w and (11) for the pair (m', w'), where $m' >_{w'} m$ and $w' >_{m'} w$ is as shown in Table 1.

This is the node-edge incidence matrix of an odd cycle; showing that the constraint matrix of (1)–(4) is not totally unimodular.

The formulation given in (9)–(12) leads to the intriguing question: Can we relax the requirement that everyone marry? More precisely, we might

Table 1

$x(m, w)$	$x(m', w)$	$x(m, w')$	
1	0	1	row (9) for man m
1	1	0	row (10) for woman w
0	1	1	row (11) for (m', w')

Table 2

Man	Preferences	Woman	Preferences
a	2, 1, 3	1	c, b, a
b	3, 1, 2	2	b, c, a
c	1, 2, 3	3	a, b, c

ask: Are the extreme point solutions of the following system integer valued?

$x(m, W) \leqslant 1$

for each man $m \in M$,   (13)

$x(M, w) \leqslant 1$

for each woman $w \in W$,   (14)

$x(m >, w) + x(m, w) + x(m, w >) \leqslant 1$

for each $(m, w) \in M \times W$,   (15)

$x(m, w) \geqslant 0$

for each $(m, w) \in M \times W$.   (16)

Kathie Cameron [4] provided the following example showing that (13)–(16) may have fractional extreme points. Equivalently, this example shows that the constraint matrix of (13)–(16) is not perfect.

Example. Consider the preference lists in Table 2. We leave it to the reader to verify that coefficients of the columns $x(a, 2)$, $x(a, 3)$, $x(b, 1)$, $x(b, 3)$ and $x(c, 2)$ in rows (13) for the men a and is b, (14) for the women 2 and 3 and (15) for the pair $(c, 1)$ correspond to a chordless odd cycle.

4. Conclusions

Although it is doubtful that any conclusions can be drawn about the relative value of monogamous versus polygamous marriage outside the context of a specific culture (my Saudi Arabian friend insists on this), we have demonstrated an equivalence between recent algorithmic approaches to the stable marriage problem and pivot operations in a structured linear program.

Our linear programming formulation is one of few examples that have integer-valued extreme points although the constraint matrix is neither

totally unimodular nor perfect. We are currently investigating extensions to the related stable roommates problem.

Acknowledgements

Appendix

Proof of Theorem 1. It is immediate that the incidence vector of each stable matching is an extreme point of (1)–(4). We prove that each extreme point of (1)–(4) is the incidence vector of a stable matching by showing that (1)–(4) is equivalent to (the dual of) a network flow problem.

Note that by Lemma 1, if the pair (m, w) is eliminated in Algorithm 1 or Algorithm 2, then for each x satisfying (1)–(4), $x(m, w) = 0$.

We construct an equivalent network flow problem over the remaining variables by induction on the number of rotations eliminated. Let $R(k)$ be the members of R, which can be identified by eliminating at most $k - 1$ rotations. The inductive hypothesis for an integer k assumes that for each $x \in \mathbb{R}^{|M| \times |W|}$ satisfying (1)–(4), there is $y \in \mathbb{R}^{|R|}$, with $y(\rho(\mu_F)) = 1$, satisfying:

(A) If (m, w) is in the rotation $\rho \in R(k)$, then $x(m, w) = y(\rho) - y(\rho_M(m, w))$. (If $\rho_M(m, w)$ does not exist, then $y(\rho_M(m, w))$ is defined to be zero.)

In Lemma 2 we proved that if $\rho = \{(m_j, w_j): j \in [1, \ldots, r]\}$ is in $R(1)$, then for each $x \in \mathbb{R}^{|M| \times |W|}$ satisfying (1)–(4),

(i) $x(m_j, w_j) = x(m_1, w_1)$ for each $j \in [2, \ldots, r]$,

(ii) $x(m_{j-1} > i > m_j, w_j) = 0$ for each $j \in [2, \ldots, r]$, and

(iii) $x(m_r > i > m_1, w_1) = 0$.

Thus, letting $y(\rho(m, w)) = x(m, w)$ for each (m, w) with $\rho(m, w) \in R(1)$, we see that y satisfies the inductive hypothesis for $k = 1$.

Assume that the inductive hypothesis holds for some $k \geq 1$ and consider a rotation $\rho =$

$\{(m_j, w_j): j \in [1, \ldots, r]\}$ in $R(k + 1)$. By the inductive hypothesis,

$$x(m_r, > w_1) = \sum \left(x(m_r, j): (m_r, j) \text{ is in a} \right.$$
$$\text{rotation and } j >_{m_r} w_1 \Big)$$
$$= \quad x(m_r, w_r) + y(\rho_M(m_r, w_r)). \quad (17)$$

Likewise,

$$x(m_r >, w_1) = \sum \left(x(j, w_1): (j, w_1) \text{ is in a} \right.$$
$$\text{rotation and } m_r >_{w_1} j \Big)$$
$$+ x(m_r > i > m_1, w_1)$$
$$= x(m_1, w_1) + y(\rho_M(m_1, w_1))$$
$$+ x(m_r > i > m_1, w_1). \quad (18)$$

Combining (17) and (18) with constraint (3) for the pair (m_r, w_1), we see that

$$x(m_1, w_1) + y(\rho_M(m_1, w_1))$$
$$+ x(m_r > i > m_1, w_1)$$
$$- x(m_r, w_r) - y(\rho_M(m_r, w_r)) \leqslant 0. \quad (19)$$

Likewise,

$$x(m_j, w_j) + y(\rho_M(m_j, w_j))$$
$$+ x(m_{j-1} > i > m_j, w_j)$$
$$- x(m_{j-1}, w_{j-1}) - y(\rho_M(m_{j-1}, w_{j-1})) \leqslant 0$$
$$\text{for each } j \in [2, \ldots, r]. \quad (20)$$

Combining (19) and (20) we see that

$$x(m_r > i > m_1, w_1) = 0,$$
$$x(m_{j-1} > i > m_j, w_j) = 0$$
$$\text{for each } j \in [2, \ldots, r],$$
$$y(\rho_M(m_1, w_1)) - y(\rho_M(m_r, w_r))$$
$$= x(m_r, w_r) - x(m_1, w_1),$$
$$y(\rho_M(m_j, w_j)) - y(\rho_M(m_{j-1}, w_{j-1})),$$
$$= x(m_{j-1}, w_{j-1}) - x(m_j, w_j)$$

for each $j \in [2, \ldots, r]$.

Finally, letting

$$y(\rho) = x(m_r, w_r) + y(\rho_M(m_r, w_r)),$$

i.e.,

$$y(\rho) = x(m_r, > w_r) + x(m_r, w_r),$$

we find that

$$x(m_j, w_j) = y(\rho) - y(\rho_M(m_j, w_j))$$

for each $j \in [1, \ldots, r]$;

proving that for each $x \in \mathbb{R}^{|M| \times |W|}$ satisfying (1)–(4), there is $y \in \mathbb{R}^{|R|}$, with $y(\rho(\mu_F)) = 1$, satisfying:

(B) If (m, w) is in the rotation $\rho \in R$, then $x(m, w) = y(\rho) - y(\rho_M(m, w))$. (If $\rho_M(m, w)$ does not exist, then $y(\rho_M(m, w))$ is defined to be zero.)

We demonstrate that the extreme points of (1)–(4) are integer valued by showing that the vectors $y \in \mathbb{R}^{|R|}$, with $y(\rho(\mu_F)) = 1$, for which the transformation defined by (B) and,

(C) $x(m, w) = 0$ if (m, w) is not in any stable matching gives a solution x to (1)–(4), are exactly the solutions to

$$y(\rho(m, w)) - y(\rho_M(m, w)) \geqslant 0$$

for each $(m, w) \in M \times W,$ (21)

$$y(\rho_M(m, w)) - y(\rho_W(m, w)) \geqslant 0$$

for each $(m, w) \in M \times W,$ (22)

$$y(\rho(\mu_F)) = 1, \quad 1 \geqslant y(\rho) \geqslant 0$$

for each $\rho \in R.$ (23)

Consider a vector $y \in \mathbb{R}^{|R|}$ and let x be defined by (B) and (C). Clearly x is non-negative if and only if y satisfies (13). Likewise, by (C),

$$x(m >, w) = \sum \left(x(i, w) : (i, w) \text{ is in a stable} \right.$$
$$\left. \text{matching and } m >_w i \right).$$

Therefore,

$$x(m >, w) = \sum \left(x(i, w) : (i, w) \text{ is in a stable} \right.$$
$$\left. \text{matching and } m >_w i \right)$$
$$= \sum \left(y(\rho(i, w)) - y(\rho_M(i, w)) \right):$$

(i, w) is in a stable matching and $m >_w i$).

Recall that if (m, w) is in a rotation, then $\rho_M(m, w) = \rho_W(m, w)$ and so, $x(m >, w) = y(\rho_W(m, w))$ and, in particular, $x(M, w) = 1$. Likewise, $x(m, > w) = y(\rho_M(m, w))$, and $x(m, W) = 1$. From which we see that x satisfies (1)–(3) if and only if y satisfies (21) and (23).

Since (21)–(23) has integer extreme points and the transformation (B) and (C) maintains integrality, the linear system (1)–(4) has integer extreme points. $\square$

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