1) This question is worth 15 out of 25 points. I gave 2 points to everybody just for handing in the assignment. Each of the 6 examples is worth 1.5 points (0.5 just for giving the correct true/false answer, and 1 point for the proof). The proof in part (b) is worth 4 points, two points for each direction.

Here are some general remarks. Many people went on to work with the old P3' statement and not the revised one. That caused them problems for the proof in part (b). Also some people involved weights and scores and so on in their proofs of part (a). It is important to understand that maximality of admissible subsets has nothing to do with maximizing those scores. Please talk to one of the TAs if you are still unclear about this.

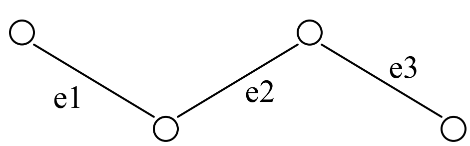
(a)
P3': For every subset U of the base set S, every maximal admissible subset of U has the same cardinality.

Example 1: Property P3' does NOT hold.
Counterexample:

n=4, S={1, 2, 3, 4}
b = 120 yen
c1=20, c2=30, c3=50, c4=70

Look at U=S. Then consider A={3, 4} with cost c3+c4 = 50+70 = 120. A is a maximal admissible subset of U since we cannot add another piece to it without exceeding the cost limit of 120 yen. Consider B={1, 2, 3} with cost c1+c2+c3 = 20+30+50 = 100. For the same reason as above, B is also a maximal admissible subset of U, yet |B|≠|A|.

Example 2: Property P3' does NOT hold.
Counterexample: consider the following graph with S={e1, e2, e3}:



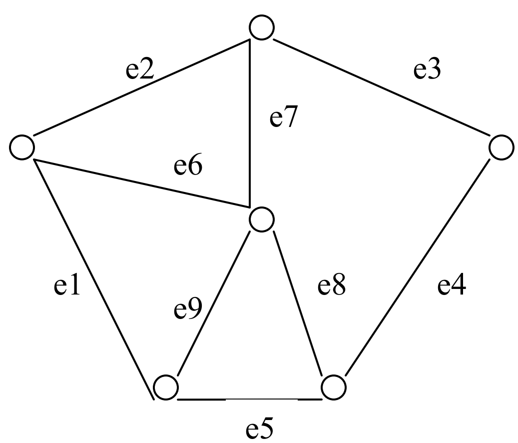
Look at U=S. Then A={e1, e3} and B={e2} are both maximal admissible subsets of U, yet |A|≠|B|.

Example 3: Property P3' DOES hold.
Proof: By contradiction. Let S be a set of n people, k a positive integer, and let U be any subset of S. If |U| ≤ k, the property holds since U itself is the only maximal admissible subset of U. Assume |U| > k and let A and B be maximal admissible subsets of U, but with different cardinality, i.e. |A|≠|B|. Without loss of generality I will assume that B is the larger of the two: |A|<|B|. Since A and B are both admissible we must have |A| ≤ k and |B| ≤ k. The last two statements can be combined to give |A| < |B| ≤ k and therefore

|A| is strictly smaller than k. In this case we can add a person to A from U\A and it will still be an admissible subset, which contradicts our assumption that A was maximal.

Example 4: Property P3' DOES hold.
Proof: Let S = {1, 2, …, n}, and let U be any subset of S. If we consider the columns of the matrix as vectors in R^m, then finding a maximal admissible subset of U amounts to finding columns that form a basis for the subspace that the columns of U span. From linear algebra we get that every linearly independent subset s of vectors of a subspace is part of some finite basis for the subspace. In our case the linearly independent subsets of vectors are our admissible sets, and the bases are our maximal admissible subsets. Two bases for the same subspace have the same number of elements.

Example 5: Property P3' does NOT hold.
Counterexample:



Let U = {e6, e7, e8}. Let A = {e6, e7}. It is admissible because it can be extended into a cycle of length 6 by adding some edges from S (e3, e4, e5, e1). It is maximal over U because adding the remaining edge, e8, gives a set of edges that cannot be part of a TSP tour. Consider B = {e8}. It is admissible since it can be extended to give a cycle of length 6 (with edges e4, e3, e2, e1, e9). However it is maximal over U because adding e6 or e7 gives a structure that can't be part of a TSP tour. The cardinalities of A and B are different, therefore P3' fails.

Example 6: Property P3' DOES hold.
Proof: Let G=(V,E) be any graph with n vertices, and let U be any subset of E. If the edges in U don't form a cycle, then the property holds since U itself is the only maximal admissible subset of U.

If U contains one or more cycles, let k be the number of connected components in U. Consider one of those connected components, call it ci, and let it have mi vertices. If ci contains no cycle, then it is a connected acyclic graph, which is the definition of a tree, and therefore it must have exactly mi-1 edges. If ci contains one or more cycles, we need to remove edges from it until we eliminate all cycles in it. Since ci is connected but cyclic, it must contains a tree, and it is possible to obtain this tree by removing some fixed number of edges. The way to remove these edges may not be unique but the number of edges to remove is, since we must obtain a tree with exactly mi-1 vertices.

So the connected components c1, c2, …, ck are reduced to trees with m1-1, m2-1, …, mk-1 edges. The total m1+m2+…+mk – k = |U| - k gives the unique size of all maximal admissible subsets of U.

(b)
If a selection problem satisfies P1, P2 and P3, then it also satisfies P3'.
Proof:
Assume P3 holds but P3' doesn't, i.e. Y1 and Y2 are maximal admissible subsets of some $A \subseteq S$, with $|Y1| \geq |Y2| + 1$. By P3 there exists $y \in Y1 \setminus Y2$ such that $Y2 \cup \{y\}$ is admissible. This contradicts the maximality of Y2.

If a selection problem satisfies P1, P2 and P3', then it also satisfies P3.
Proof:
Assume P3' holds but P3 doesn't. Let U and V be admissible subsets of S with $|V| = |U|+1$. The exchange property doesn't hold so there doesn't exist $e \in V \setminus U$ such that $U \cup \{e\}$ is admissible. Then U must be maximal in $A = U \cup V$. By P3' the maximal admissible subsets of A have the same cardinality. V is not necessarily maximal in A, but there exists W such that $V \subseteq W$ and W is maximal in A. Then U and W are both maximal in A but have different cardinality.