All Meals for a Dollar and other vertex enumeration problems

David Avis

April 5, 2012

Hot news: Lovasz wins Kyoto Prize

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The Kyoto prize / Laureates / The 2010 Kyoto Prize Laureates

Advanced Technology / Prize Field: Biotechnology and Medical Technology

Shinya Yamanaka

Japan / September 4, 1962
Medical Scientist
Professor, Kyoto University

| Profile | Achievements | Press page |

"Development of Technology for Generating Induced Pluripotent Stem (iPS) Cells"

By introducing just four transcription factor genes into dermal fibroblasts, Dr. Yamanaka succeeded in producing induced pluripotent stem (iPS) cells, which exhibit a pluripotency similar to that of embryonic stem (ES) cells. The iPS cell technology is now expected not only to expand the possibilities of regenerative medicine, but also to make significant contributions to the rapid progress of medical science in general.

Basic Sciences / Prize Field: Mathematical Sciences (including Pure Mathematics)

László Lovász

Hungary, U.S.A. / March 9, 1948
Mathematician
Professor, Eötvös Loránd University

| Profile | Achievements | Press page |

"Outstanding Contributions to Mathematical Sciences Based on Discrete Optimization Algorithms"

Through his advanced research on discrete structures, Dr. Lovász has provided a link among various branches of mathematics in terms of algorithms, thereby influencing a broad spectrum of the mathematical sciences - including discrete mathematics, combinational optimization and theoretical computer science. In so doing, Dr. Lovász has made outstanding contributions to the advancement of both the academic and technological possibilities of the mathematical sciences.

Arts and Philosophy / Prize Field: Arts (Painting, Sculpture, Craft, Architecture, Design)

Modelling and optimization

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- The decisions are modelled as decision variables
- The constraints and the objective are stated in terms of the decision variables
- If the constraints and objective are linear functions, it is called a linear program

Diet problem

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- Constraints: There are minimum daily requirements for calories, vitamins, calcium, etc. There is a maximum amount of each food you can eat.
- Objective Eat for less than $1.

Sample data

	Food	Serv. Size	Energy (kcal)	Protein (g)	Calcium (mg)	Price ¢	Max Serv.
x_1	Oatmeal	28g	110	4	2	3	4
x_2	Chicken	100g	205	32	12	24	3
x_3	Eggs	2 large	160	13	54	13	2
x_4	Milk	237ml	160	8	285	9	8
x_5	Cherry Pie	170g	420	4	22	20	2
x_6	Pork w. beans	260g	260	14	80	19	2
	Min. Daily Amt.		2000	55	800		

The decision variables are $x_1, x_2, \ldots, x_6$.

Fractional servings are allowed.

From *Linear Programming*, Vasek Chvátal, 1983

Linear programming formulation for diet problem

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$$
\begin{aligned}
\min z &= 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \\
\text{s.t.} \quad & 110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 \geq 2000 \\
& 4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 \geq 55 \\
& 2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 \geq 800 \\
& 0 \leq x_1 \leq 4, \quad 0 \leq x_2 \leq 3, \quad 0 \leq x_3 \leq 2, \\
& 0 \leq x_4 \leq 8, \quad 0 \leq x_5 \leq 2, \quad 0 \leq x_6 \leq 2
\end{aligned}
$$

General linear programming problem

$$
\begin{aligned}
max\ z \ = \ & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
s.t. \quad & a_{11}x_1 + a_{12}x_2 + a_{1n}x_n \leq b_1 \\
& a_{21}x_1 + a_{22}x_2 + a_{2n}x_n \leq b_2 \\
& \ldots \\
& a_{m1}x_1 + a_{m2}x_2 + a_{mn}x_n \leq b_m \\
& x_1 \geq 0,\ x_2 \geq 0,\ \ldots,\ x_n \geq 0
\end{aligned} \tag{1}
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- $c_1,\ c_2,\ \ldots,\ c_n,\ b_1,\ b_2,\ \ldots, b_m$ and $a_{11},\ \ldots,\ a_{ij},\ \ldots, a_{mn}$ are input data
- The constraints (1) define a *convex polyhedron*

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- "In terms of widespread application, Dantzig's algorithm is one of the most successful of all time: Linear programming dominates the world of industry..." (Top 10 Algorithms of the 20th century)
- It gave rise to the field of Operations Research (OR).

Operations Research faculty at Stanford (1969)

George Dantzig is on the far left, then Alan Manne, Frederick Hillier, Donald Iglehart, Arthur Veinott Jr., Rudolf E. Kalman, Gerald Lieberman, Kenneth Arrow and Richard Cottle.

Sensei and Seito

Vasek Chvátal

Another OR graduate from Stanford

Hatoyama file:///C:/cygwin/home/avis/talks/allmeals/hatoy

informs online
Institute for Operations Research and the Management Sciences

In The Media

PM with OR degree steps down

"Japan's former prime minister, Yukio Hatoyama, could not apply math modeling to solving two pressing political problems......."

"Before entering politics, Hatoyama in the 1970s received a Ph.D in engineering in a field called operations research, which employs applied mathematics to solve complex problems, at Stanford University."

Linear programming solution

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- $x_1 = 4$(oatmeal) $x_4 = 4.5$(milk) $x_5 = 2$(pie) cost=92.5 ¢

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- $x_1 = 4$(oatmeal) $x_4 = 4.5$(milk) $x_5 = 2$(pie) cost=92.5 ¢
- Where are the chicken, eggs and pork?
- Do I have to eat the same food every day?

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- We obtained a unique optimum solution, but ...
- ... people (and managers) like to make choices!

Ask the right question!

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- Q: What are all the meals I can eat for at most \$1?
- A: An infinite number! Add any small amount .....
- Q: Can you give me some different meals at least?
- A: Yes! In fact I can describe all allowable meals for under \$1

All meals for a dollar

Any solution to these inequalities is a meal for under $1:

$$\begin{aligned}
3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 &\leq 100 \\
110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 &\geq 2000 \\
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\end{aligned}
$$

- But this is just a restatement of the problem ....
- ... how do I find these solutions?

A more useful solution

All menus for a $1

All (17) Extreme
Solutions to the Diet Problem with Budget $1.00

Cost	Oat-meal	Chicken	Eggs	Milk	Cherry Pie	Pork Beans
92.5	4.	0	0	4.5	2.	0
97.3	4.	0	0	8.	0.67	0
98.6	4.	0	0	2.23	2.	1.40
100.	1.65	0	0	6.12	2.	0
100.	2.81	0	0	8.	0.98	0
100.	3.74	0	0	2.20	2.	1.53
100.	4.	0	0	2.18	1.88	1.62
100.	4.	0	0	2.21	2.	1.48
100.	4.	0	0	5.33	2.	0
100.	4.	0	0	8.	0.42	0.40
100.	4.	0	0	8.	0.80	0
100.	4.	0	0.50	8.	0.48	0
100.	4.	0	1.88	2.63	2.	0
100.	4.	0.17	0	2.27	2.	1.24
100.	4.	0.19	0	8.	0.58	0
100.	4.	0.60	0	3.73	2.	0
100.	4.	0	1.03	2.21	2.	0.78

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- Taking convex combinations of rows gives new meals
- Eg. Taking half each of the last two rows gives a $1 meal with all foods

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$$x = \sum_{i=1}^{N} \lambda_i v_i$$

$$\text{where} \quad \sum_{i=1}^{N} \lambda_i = 1, \quad \lambda_i \geq 0, \; i = 1, 2, ..., N$$

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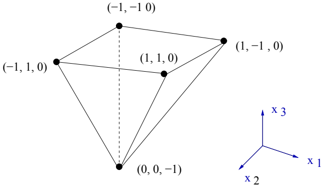
- Vertex enumeration: H-representation $\Rightarrow$ V-representation
- Convex hull problem: V-representation $\Rightarrow$ H-representation

Example in R^3



H-representation:

$$\begin{aligned} 1 - x_1 \quad\quad + x_3 &\geq 0 \\ 1 \quad\quad - x_2 + x_3 &\geq 0 \\ 1 + x_1 \quad\quad + x_3 &\geq 0 \\ 1 \quad\quad + x_2 + x_3 &\geq 0 \\ - x_3 &\geq 0 \end{aligned}$$

V-representation:

$$v_1 = (-1,1,0), \quad v_2 = (-1,-1,0), \quad v_3 = (1,-1,0),$$

$$v_4 = (1,1,0), \quad v_5 = (0,0,-1)$$

Dictionary representation

Introduce slack variables:

$$\begin{aligned}
x_4 &= 1 - x_1 \quad + x_3 \geq 0\\
x_5 &= 1 \quad - x_2 + x_3 \geq 0\\
x_6 &= 1 + x_1 \quad + x_3 \geq 0\\
x_7 &= 1 \quad + x_2 + x_3 \geq 0\\
x_8 &= \quad - x_3 \geq 0
\end{aligned}$$

$B = \{4, 5, 6, 7, 8\}$, indices of *basic* variables.

$N = \{1, 2, 3\}$, indices of *co-basic* variables.

The dictionary with $N = \{4, 5, 8\}$ is *feasible* and represents the vertex $(1, 1, 0)$:

$$\begin{aligned}
x_1 &= 1 - x_4 \quad - x_8\\
x_2 &= 1 \quad - x_5 - x_8\\
x_3 &= \quad - x_8\\
\hline
x_6 &= 2 - x_4 \quad - 2x_8\\
x_7 &= 2 \quad - x_5 - 2x_8
\end{aligned}$$

Note: x_1, x_2, x_3 are basic and $x_6 \geq 0, \;\; x_7 \geq 0$

Adjacency: Pivoting

$$\begin{aligned} x_1 &= 1 - x_4 \quad\quad - x_8 \\ x_2 &= 1 \quad\quad - x_5 - x_8 \\ x_3 &= 0 \quad\quad\quad\quad - x_8 \\ \hline x_6 &= \mathbf{2} - x_4 \quad\quad - 2x_8 \\ x_7 &= \mathbf{2} \quad\quad - x_5 - 2x_8 \end{aligned}$$

B = { *1, 2, 3,* ***6, 7*** }

N = { ***4, 5 8*** }

Entering variable: ***4***

Leaving variable: ***6***

$$\begin{aligned} x_1 &= -1 \quad\quad + x_6 + x_8 \\ x_2 &= 1 - x_5 \quad\quad - x_8 \\ x_3 &= 0 \quad\quad\quad\quad - x_8 \\ \hline x_4 &= \mathbf{2} \quad\quad - x_6 - 2x_8 \\ x_7 &= \mathbf{2} - x_5 \quad\quad - 2x_8 \end{aligned}$$

B = { 1, 2, 3, ***4, 7*** }

N = { ***5, 6, 8*** }

x4 = 2 >= 0

x5 = 2 >= 0

Feasible Pivot!

Graph search for vertex enumeration

- Adjacency oracle for a feasible dictionary N:

$$Adj(N,i,j) = \begin{cases} \emptyset & N \cup \{i\}/\{j\} \text{ infeasible dictionary} \\ \emptyset & x_j \text{ has zero coefficient in row } x_i \\ N \cup \{i\}/\{j\} & \text{feasible dictionary} \end{cases}$$

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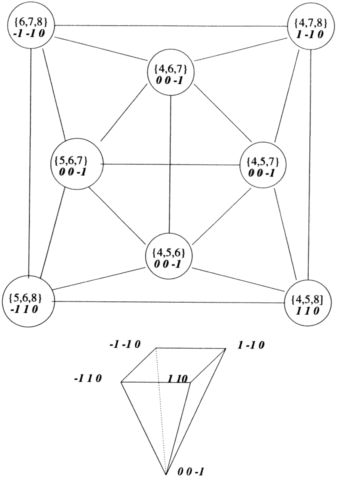
- Eg. DFS or BFS

Graph of all feasible dictionaries



Problems with DFS/BFS algorithms

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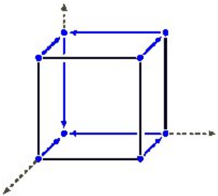
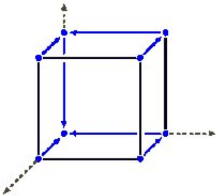
- Often we just want to examine vertices, not keep them all
- Reverse search allows this to be done in $O(mn)$ space, ie. input size.

Reverse search algorithm Avis-Fukuda 1991



Reverse search algorithm Avis-Fukuda 1991



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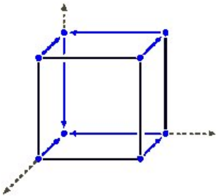

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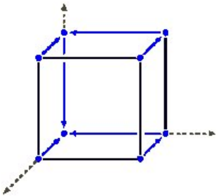
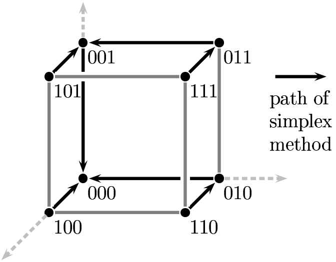


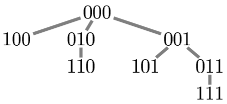
- Simplex method gives a path from any vertex to the optimum
- The set of all such paths is a spanning tree of the polyhedron
- Reverse search builds this tree starting at the origin, *reversing* the simplex method

Reverse search algorithm

http://cgm.cs.mcgill.ca/ avis/C/lrs.html



(a) The "simplex tree" induced by the objective $(-\sum x_i)$.



(b) The corresponding reverse search tree.