Modelling and Optimization: All Meals for a Dollar

June 16, 2010

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- The decisions are modelled as decision variables
- The constraints and the objective are stated in terms of the decision variables
- If the constraints and objective are linear functions, it is called a linear program

Diet problem

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- Objective Eat for less than $1.

Sample data

	Food	Serv. Size	Energy (kcal)	Protein (g)	Calcium (mg)	Price (cents)	Max Serv.
x_1	Oatmeal	28g	110	4	2	3	4
x_2	Chicken	100g	205	32	12	24	3
x_3	Eggs	2 large	160	13	54	13	2
x_4	Whole Milk	237cc	160	8	285	9	8
x_5	Cherry Pie	170g	420	4	22	20	2
x_6	Pork w. beans	260g	260	14	80	19	2
	Min. Daily Amt.		2000	55	800		

The decision variables are $x_1, x_2, ..., x_6$.
Fractional servings are allowed.
From *Linear Programming* , V. Chvátal, 1983

Linear programming formulation for diet problem

$$
\begin{array}{ll}
\text{minimize} & 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \\
\text{subject to} & 0 \le x_1 \le 4 \\
& 0 \le x_2 \le 3 \\
& 0 \le x_3 \le 2 \\
& 0 \le x_4 \le 8 \\
& 0 \le x_5 \le 2 \\
& 0 \le x_6 \le 2
\end{array}
$$

$$
\begin{aligned}
110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 &\ge 2000 \\
4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 &\ge 55 \\
2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 &\ge 800
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General linear programming problem

$$
\begin{aligned}
max\ z \ = \ & c_1x_1 + c_2x_2 + \dots + c_nx_n \\
s.t. \quad & a_{11}x_1 + a_{12}x_2 + a_{1n}x_n \leq b_1 \\
& a_{21}x_1 + a_{22}x_2 + a_{2n}x_n \leq b_2 \\
& \dots\dots \\
& a_{m1}x_1 + a_{m2}x_2 + a_{mn}x_n \leq b_m \\
& x_1 \geq 0,\ x_2 \geq 0,\ \dots,\ x_n \geq 0
\end{aligned}
$$

- $x_1,\ x_2,\ \dots,\ x_n$ are the decision variables

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$$

- $x_1,\ x_2,\ \ldots,\ x_n$ are the decision variables
- $c_1,\ c_2,\ \ldots,\ c_n,\ b_1,\ b_2,\ \ldots, b_m$ and $a_{11},\ \ldots,\ a_{ij},\ \ldots, a_{mn}$ are input data

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- It gave rise to the field of Operations Research (OR).

Operations Research faculty at Stanford (1969)

George Dantzig is on the far left, then Alan Manne, Frederick Hillier, Donald Iglehart, Arthur Veinott Jr., Rudolf E. Kalman, Gerald Lieberman, Kenneth Arrow and Richard Cottle.

Sensei and Seito

Vasek Chvátal

Another OR graduate from Stanford

informs online
Institute for Operations Research and the Management Sciences

In The Media

 PM with OR degree steps down

"Japan's former prime minister, Yukio Hatoyama, could not apply math modeling to solving two pressing political problems......."

"Before entering politics, Hatoyama in the 1970s received a Ph.D in engineering in a field called operations research, which employs applied mathematics to solve complex problems, at Stanford University."

Linear programming solution

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- $x_1 = 4, x_2 = 4.5, x_6 = 2$. Cost is 92.5 cents.
- Where are the chicken, eggs and pork?
- Do I have to eat the same food every day?

Ask the right question!

- Q: What are all the meals I can eat for at most $1?

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- Q: What are all the meals I can eat for at most \$1?
- A: An infinite number! Add any small amount .....
- Q: Can you give me some different meals at least?
- A: Yes! In fact I can describe all possible meals for under \$1

Any solution to these inequalities is a possible meal

All Meals for a Dollar

$$3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \leq 100$$

$$0 \leq x_1 \leq 4$$

$$0 \leq x_2 \leq 3$$

$$0 \leq x_3 \leq 2$$

$$0 \leq x_4 \leq 8$$

$$0 \leq x_5 \leq 2$$

$$0 \leq x_6 \leq 2$$

$$110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 \geq 2000$$

$$4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 \geq 55$$

$$2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 \geq 800$$

Vertex Enumeration Problem:
Compute all vertices of this polyhedron.

A more useful solution

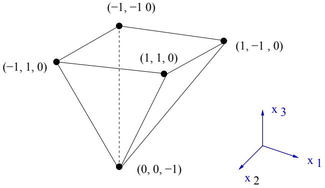
All menus for a $1

All (17) Extreme

Solutions to the Diet Problem with Budget $1.00

Cost	Oat-meal	Chicken	Eggs	Milk	Cherry Pie	Pork Beans
92.5	4.	0	0	4.5	2.	0
97.3	4.	0	0	8.	0.67	0
98.6	4.	0	0	2.23	2.	1.40
100.	1.65	0	0	6.12	2.	0
100.	2.81	0	0	8.	0.98	0
100.	3.74	0	0	2.20	2.	1.53
100.	4.	0	0	2.18	1.88	1.62
100.	4.	0	0	2.21	2.	1.48
100.	4.	0	0	5.33	2.	0
100.	4.	0	0	8.	0.42	0.40
100.	4.	0	0	8.	0.80	0
100.	4.	0	0.50	8.	0.48	0
100.	4.	0	1.88	2.63	2.	0
100.	4.	0.17	0	2.27	2.	1.24
100.	4.	0.19	0	8.	0.58	0
100.	4.	0.60	0	3.73	2.	0
100.	4.	0	1.03	2.21	2.	0.78

Example in R^3



H-representation:

$$\begin{aligned} 1 - x_1 \quad\quad + x_3 &\geq 0 \\ 1 \quad\quad - x_2 + x_3 &\geq 0 \\ 1 + x_1 \quad\quad + x_3 &\geq 0 \\ 1 \quad\quad + x_2 + x_3 &\geq 0 \\ - x_3 &\geq 0 \end{aligned}$$

V-representation:

$$v_1 = (-1, 1, 0), \quad v_2 = (-1, -1, 0), \quad v_3 = (1, -1, 0),$$

$$v_4 = (1, 1, 0), \quad v_5 = (0, 0, -1)$$

Convex Hull and Vertex Enumeration

A convex polyhedron P in R^d has two representations:

H-representation:

A set of m *facet* generating inequalities.

$$P = \{x \in R^d \mid b + Ax \geq 0\}$$

V-representation:

A set of *vertices* $v_1, \cdots, v_s$ and *extreme rays* $z_1, \cdots, z_u$.

$$P = \{x \in R^d \mid x = \sum_{i=1}^{s} \lambda_i y_i + \sum_{j=1}^{u} \mu_j z_j,$$

$$\lambda_i \geq 0, \mu_j \geq 0, \sum_{i=1}^{s} \lambda_i = 1\}.$$

Vertex Enumeration Problem:

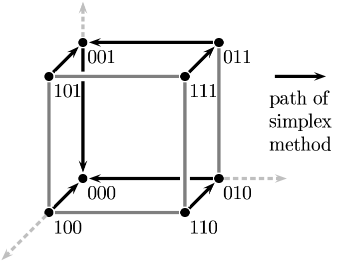
H-representation => V-representation

Facet Enumeration Problem:

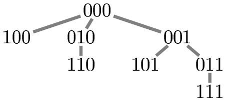
V-representation => H-representation

Reverse search algorithm

http://cgm.cs.mcgill.ca/ avis/C/lrs.html



(a) The "simplex tree" induced by the objective $(-\sum x_i)$.



(b) The corresponding reverse search tree.