**Computational Intractability** 2010/5/20, Revised 2011/6/8 DA

# Lecture 6

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We know how to solve linear programs and prove solutions are correct. How can we use this to solve integer programs? The first approach is to try solving it as an LP. In this lecture we consider a multi-period scheduling problem called the Uncapacitated Lot Size (ULS) problem.

## 1 ULS problem with no fixed costs

In this problem we have $n$ periods and a demand at each period. The problem is to schedule the production at each of $n$ periods to meet the demand of that period, and possibly leave some extra inventory for the periods to come. There is a given production cost per unit, and a holding cost for the inventory. For simplicity, we will assume the holding cost is \$1/unit per time period. A flow-chart for the problem with $n = 3$ periods is given in Figure 1.

Symbols:

1. n : periods
2. $d_i$: demand at period i
3. $p_i$: cost of producing 1 unit in period i
4. $x_i$: amount produced in period i
5. $s_i$ : amount of inventory at the end of period i.

Although we are interested only in integer solutions (imagine we are producing airplanes) we can will formulate the problem in Figure 1 as a linear program called a LP-relaxation:

$$\begin{aligned}
&min : z = 3x_1 + 4x_2 + 3x_3 + s_1 + s_2 + s_3 && (1)\\
&x_1 = 6 + s_1\\
&x_2 = 7 - s_1 + s_2\\
&x_3 = 4 - s_2 + s_3\\
&x_i, s_i \geq 0, \quad i = 1, 2, 3.
\end{aligned}$$

The input file for lp_solve is *suls3.lp*. The solution by lp_solve is $x_1 = 13, x_3 = 4$ with $z = 58$. Since we have an integer solution, we are done. We were lucky, or were we? If we change the input data, do we always get an integer solution?

In general we may ask when does an LP-relaxation of an ILP have an integer solution? This is a difficult question in general, as it often depends on the objective function. In

Figure 2, for some objective functions we get an integer solution, but for other objective functions we will not. One thing we can say is that an LP has an integer solution for every objective function if and only if all vertices of the feasible region are integer.

- **Facts:**

1. An optimum LP solution can always be found at a vertex (Simplex method gives it)
2. If any vertex v is fractional, there is always some objective function optimizing here uniquely.

Proof: Consider any vertex $v$ and write down its dictionary (see Figure 3). Assuming we have a maximization problem, we can replace the current objective row with

$$w = 0 - \sum_{j \in N} x_j \qquad N = \text{cobasis for } v.$$

We can rewrite $w$ by substituting for any slack variable in $N$ using the initial dictionary which defines the slacks. We now have $w$ in terms of original decision variables, and this function uniquely optimizes at vertex $v$. This completes the proof.

In order to check if all vertices are integer, we can use vertex enumeration to generate all vertices of a polyhedron $Ax \leq b$. There are several programs, one is lrs (others are cdd, porta,...) The input is called an H-representation (halfspace or inequality representation) and the output is called a V-representation (vertices and rays).

The input format for lrs is $[b - A]$, and equations are specified by the 'linearity' command. The lrs input file *suls3.ine* for our LP is:

- **H-representation:**

$$\begin{array}{rrrrrrrr}
\text{linearity} & 3 & 1 & 2 & 3 & & & \\
\text{vars} & b & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 \\
 & 6 & -1 & 0 & 0 & 1 & 0 & 0 \\
 & 7 & 0 & -1 & 0 & -1 & 1 & 0 \\
 & 4 & 0 & 0 & -1 & 0 & -1 & 1 \\
 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

The linearity command says there are three equations, in rows 1,2, and 3. The remaining inequalities are non-negativity constraints

The output after running lrs will be a V-representation(Vertices and Rays) of the polyhedron (see Figure 5).

The format is:

1 $x_1$ $x_2$ $\cdots$ $x_{n-1}$ vertex($x_1$, $x_2$, $\cdots$,$x_{n-1}$)<br>
0 $y_1$ $y_2$ $\cdots$ $y_{n-1}$ ray($y_1$, $y_2$, $\cdots$, $y_{n-1}$)

For our example, the output consists of 4 vertices and 4 rays:

- **V-representation:**

```
    x1  x2  x3  s1  s2  s3

1   6   7   4   0   0   0
0   0   0   1   0   0   1
1   13   0   4   7   0   0
0   0   0   1   0   0   1
1   6   11   0   0   4   0
0   0   1   0   0   1   1
1   17   0   0   11   4   0
0   1   0   0   1   1   1
```

We observe that all vertices are integers. Therefore, regardless of the objective function we will always get an integer solutions to the LP. This can be proven formally for this simple version of the ULS problem.

## 2 ULS problem with fixed costs

Whenever $x_i > 0$, we add fixed cost $f_i$. If you produce anything at all in period i, you must pay fixed cost $f_i$.

Let

$$y_i = \begin{cases} 1 & \text{produce in period i} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

We need to link $x_i$ with $y_i$. Let $M$ be any upper bound on $x_i$. The linking equation is $x_i \leq My_i$. This means that if $x_i$ is positive, then $y_i = 1$. Note that $y_1 = 1, x_1 = 0$ is feasible but it is not going to be optimum since the fixed costs are positive (ie. $y_1 = 0$ is also feasible at lower cost.)

We have the upper bounds $x_1 \leq 17$(total demand), $x_2 \leq 11$, $x_3 \leq 4$, and the new inequalities are:

$$\begin{aligned} x_1 &\leq 17y_1 \\ x_2 &\leq 11y_2 \\ x_3 &\leq 4y_3 \end{aligned} \tag{3}$$

Suppose $f_1 = 12, f_2 = 10, f_3 = 10$. The objective function now becomes

$$min : 3x_1 + 4x_2 + 3x_3 + s_1 + s_2 + s_3 + 12y_1 + 10y_2 + 12y_3 \tag{4}$$

The new input file is *luls3.lp*. If we now solve this ULS problem as an LP we find that we get an optimum solution:

$$x_1 = 13, \quad x_3 = 4, \quad s_1 = 7, \quad y_1 = 13/17, \quad y_3 = 1, \quad z = 77\ 3/17$$

which is not integer! If we round up $y_1$ to one then we get $z = 82$. This is not optimum as we can verify by adding the integer constraint, as in file *uls3.lp*. We now obtain the integer optimum solution:

$$x_1 = 17, \qquad s_1 = 14, \quad s_2 = 4, \quad y_1 = 1, \quad z = 78 \tag{5}$$

which involves only producing in period 1 then paying storage costs for periods 2 and 3. Indeed, this problem has many fractional vertices, as you will discover if you run lrs on the corresponding H-representation in file *luls3.ine*.

## 3 LP-relaxations for an Integer Program

In general integer programs have many formulations as linear programs. The property we would like is that the only feasible integer solutions to the LP-formulation are feasible integer solutions to our original problem. These formulations are called LP-relaxations (see Figure 6).

**Definition 1.** *Let $X$={feasible integer solutions to the problem} and $P = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$ be a polyhedron. $P$ is an LP-relaxation for $X$ if $X = P \cap Z^n$, where $Z^n$ are the integer vectors or length $n$.*

If we have an integer optimum solution to an LP-relaxation, then we are sure that it is the optimum integer solution of the original problem. We call the LP-relaxation and ideal formulation whenever all of its vertices are integer. In this case the LP solution is always the integer optimum solution.

## 4 Another formulation for the ULS problem with fixed costs

We introduce new decision variables. For each pair of periods $i \leq j$ we let $w_{ij}$ denote the production at period $i$ that satisfies demand in period $j$. In this model we do not need variables for the inventory between periods. The cost of inventory will be included in the coefficient $c_{ij}$ of $w_{ij}$ in the objective. Figure 4 shows the revised flowchart. The cost coefficients are given by

$$c_{ii} \ = \ p_i, \quad c_{it} = p_i + s_i + s_{i+1} + ... + s_{t-1} \quad i < t \leq n.$$

Our new LP formulation *uls3a.lp* is as follows:

$$min : 3w_{11} + 4w_{12} + 5w_{13} + 4w_{22} + 5w_{23} + 3w_{33} + 12y_1 + 10y_2 + 12y_3$$

$$\begin{array}{lll}
w_{11} = 6, & w_{12} + w_{22} = 7, & w_{13} + w_{23} + w_{33} = 4 \\
w_{11} \le 6y_1, & w_{12} \le 7y_1, & w_{13} \le 4y_1 \\
w_{22} \le 7y_2, & w_{23} \le 4y_2, & w_{33} \le 4y_3 \\
y_1 \le 1, & y_2 \le 1, & y_3 \le 1
\end{array}$$

This LP turns out to give the optimum integer solution (5) we obtained before.

Lucky? Not really! This turns out to be a perfect formulation for the ULS problem with fixed costs. This can be verified by running *lrs* on the input file *uls3a.ine*. The feasible region has only integer vertices, so we can solve it with the simplex method. In fact this formulation of the ULS problem always has only integer vertices if the input demands are integers. This was proved by Wolsey in "Uncapacitated Lot-Sizing Problems with Start-Up Costs", Operations Research (1988) 37, No. 5, pp. 741-747.

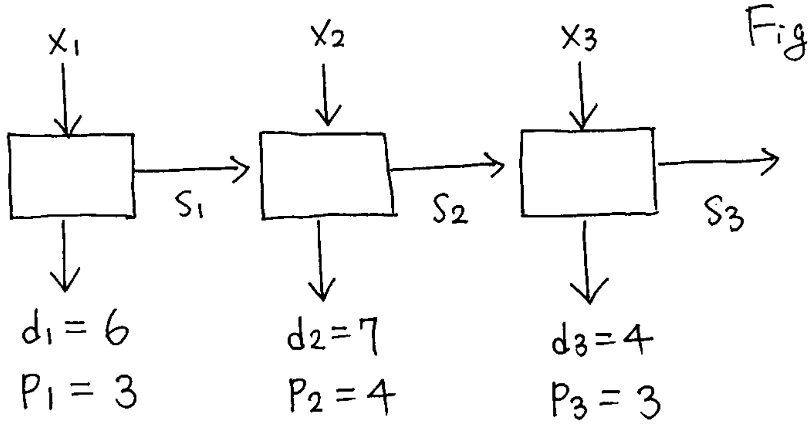


Figure 1

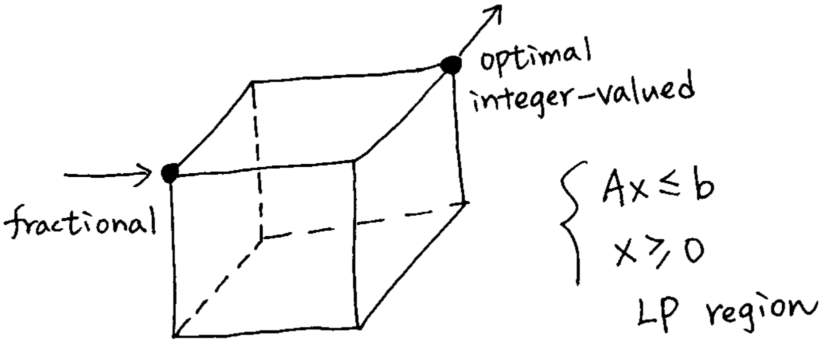


Figure 2

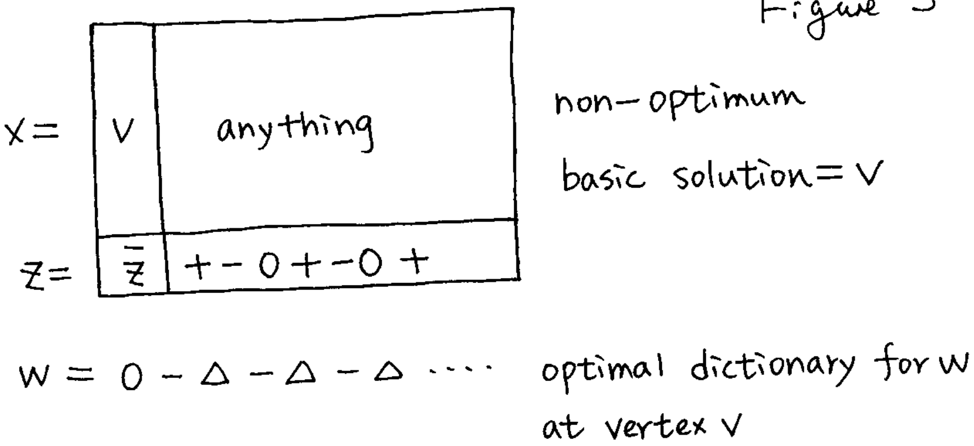
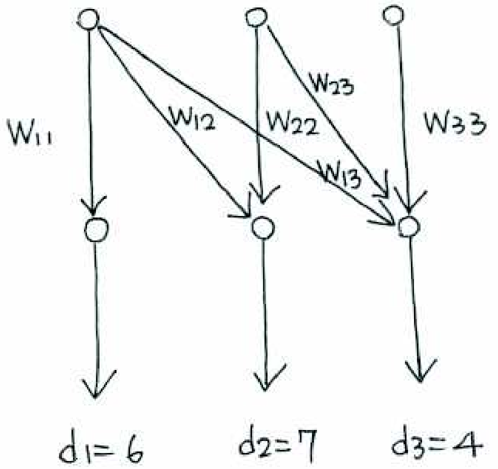


Figure 3

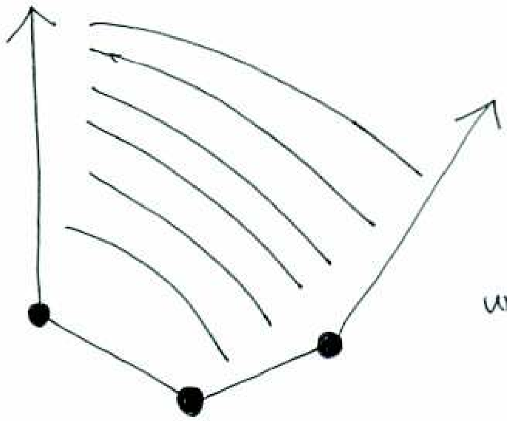
Figure 4



$W_{ij}$ : amount produced in period $i$ and used in period $j$

include storage cost for $j > i$

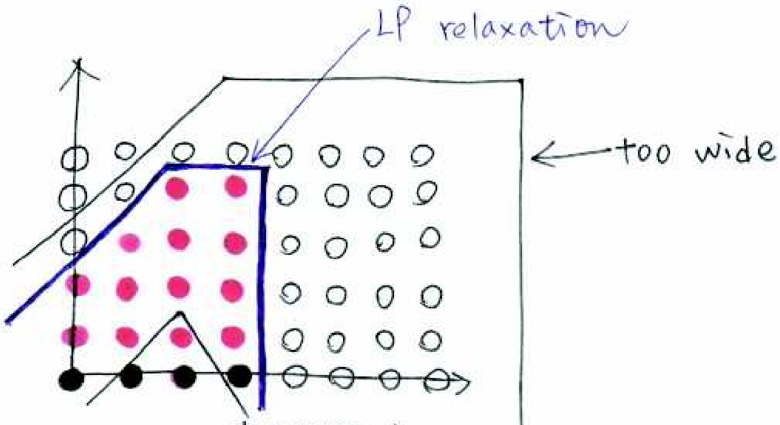
Figure 5



$\underline{A}\underline{x} \le \underline{b}$

unbounded : 3 vertices
2 rays.

Figure 6


LP relaxation
too wide
too narrow