Uncapacitated Lot-Sizing Problems with Start-Up Costs
Author(s): Laurence A. Wolsey
Source: *Operations Research*, Vol. 37, No. 5 (Sep. - Oct., 1989), pp. 741-747
Published by: INFORMS
Stable URL: http://www.jstor.org/stable/171019
Accessed: 12/06/2013 03:03

# UNCAPACITATED LOT-SIZING PROBLEMS WITH START-UP COSTS

**LAURENCE A. WOLSEY**

*Université Catholique de Louvain, Louvain-la-Neuve, Belgium*

(Received February 1988; revision received July 1988; accepted August 1988)

We consider the uncapacitated economic lot-sizing problem with start-up costs as a mixed integer program. A family of strong valid inequalities is derived for the class as well as a polynomial separation algorithm. It is then shown how equivalent, or possibly stronger, formulations are obtained by the introduction of auxiliary variables. Finally, some limited computational results for a single item model and a multi-item model with changeover costs are reported.

Many practical problems involve start-up costs when switching on a machine, or changeover costs when changing between items or modes of production. Here we begin to study how such situations can be effectively modeled by mixed integer programming.

We consider the uncapacitated economic lot-sizing (ULS) problem with an additional start-up cost $f_i$ if a setup cost is incurred in period $i$ and not in period $i - 1$. A standard mixed integer programming formulation of this problem is

$$\min\left\{\sum_{i=1}^{T} p_i y_i + \sum_{i=1}^{T} h_i s_i + \sum_{i=1}^{T} c_i x_i + \sum_{i=1}^{T} f_i z_i : (y, s, x, z) \in X\right\} \quad (1)$$

where $X$ is described by

$$s_{i-1} + y_i = d_i + s_i \quad \text{for } i = 1, \ldots, T$$

with

$$s_T = 0$$

$$y_i \leq d_{iT} x_i \quad \text{for } i = 1, \ldots, T$$

$$\min\{x_i, 1 - x_{i-1}\} \geq z_i \geq x_i - x_{i-1} \quad \text{for } i = 1, \ldots, T$$

$$s, y \geq 0, \quad x, z \in \{0, 1\}$$

where $d_{st}$ denotes $\sum_{i=s}^{t} d_i$, $d_i \geq 0$ are the demands, $p_i$, $h_i$, $c_i$ and $f_i$ the production, storage, setup and start-up costs in period $i$, and $y_i$, $s_i$, $x_i$, $z_i$ are the production, storage, setup and start-up variables, respectively. We assume that $x_0$ is known and that $s_0 = 0$.

Problems in which (1) appears as a relaxation have been studied by Van Wassenhove and Vanderheust (1983), Karmarkar and Schrage (1985) and Fleischmann (1987). There is an $O(T^2)$ dynamic programming algorithm for (1). Based on this algorithm, Eppen and Martin (1987) have provided a tight mixed integer programming formulation that differs from those described below.

In Section 2, we derive an exponential class of strong valid inequalities for $X$ and a polynomial separation algorithm for the resulting polytope $P$ that can be used in a cutting plane algorithm. In Section 3, we derive a formulation (or polytope $Q$) that is equivalent to $P$ that is described by a polynomial number of variables and constraints. In Section 4, we derive two other formulations that are at least as strong as the previous two. Finally, we report on some limited computational results for a single item model and a multi-item model with changeover costs.

## 1. STRONG VALID INEQUALITIES

The class of valid inequalities that we derive is closely related to the $(l, S)$ inequalities derived for the ULS model (see Barany, Van Roy and Wolsey 1984a).

**Proposition 1.** *Let $L = \{1, \ldots, l\}$ with $l \leq T$, and $S \subseteq L$ with $S$ composed of $r$ disjoint subintervals $\{S_k\}_{k=1}^{r}$ where $S_k = \{\sigma_k, \sigma_k + 1, \ldots, \tau_k\}$ with $\sigma_k \leq \tau_k$ for $k = 1, \ldots, r$. Then*

$$\sum_{j \in S} y_j \leq \sum_{k=1}^{r}\left[d_{\sigma_k l} x_{\sigma_k} + \sum_{i=\sigma_k+1}^{\tau_k} d_{il} z_i\right] + s_l \quad (2)$$

*is a valid inequality for X.*

**Proof.** Consider a point $(y^*, s^*, x^*, z^*) \in X$. If $x_j^* = 0$ for all $j \in S$, $\sum_{j \in S} y_j^* = 0$, and as $s_l^* \geq 0$, the inequality is valid. Otherwise, let $t = \arg\min\{j: j \in S, x_j^* = 1\}$.

*Subject classifications:* Programming, integer: cutting plane/facet generation algorithms. Inventory/production: scale diseconomies, lot-sizing.



0030-364X/89/3705-0741 $01.25

**Case 1.** $t = \sigma_k$ for some $k$. Then $\sum_{j\in S} y_j^* \leq \sum_{j=t}^{l} y_j^* \leq d_{tl} + s_l^* = d_{tl}x_t^* + s_l^*$, and hence, the inequality is satisfied.

**Case 2.** $t \in \{\sigma_k + 1, \ldots, \tau_k\}$ for some $k$. Then by definition of $t$, $t - 1 \in S_k \subseteq S$, and hence, $x_{t-1}^* = 0$. As $x_t^* = 1$, this implies $z_t^* = 1$. Therefore, $\sum_{j\in S} y_j^* \leq \sum_{j=t}^{l} y_j^* \leq d_{tl} + s_l^* = d_{tl}z_t^* + s_l^*$, and hence, the inequality holds.

Note that by substituting for the stock variable $s_l$, we obtain the alternative representation for the inequality (2)

$$\sum_{j\in L\setminus S} y_j + \sum_{k=1}^{r}\left[d_{\sigma_k l}x_{\sigma_k} + \sum_{i=\sigma_k+1}^{\tau_k} d_{il}z_i\right] \geq d_{1l}. \qquad (3)$$

In addition, it is not difficult to show, using similar arguments to those used in Barany, Van Roy and Wolsey (1984b), that almost all the inequalities (2) or (3) define facets of conv($X$).

Next we describe a separation algorithm for the inequalities (2) that can be used in a cutting plane algorithm.

### A Separation Algorithm for ($y^*$, $s^*$, $x^*$, $z^*$) and the Inequalities (2)

We assume that $z_i^* \leq x_i^*$ for $i = 1, \ldots, n$. For $l = 1, \ldots, T$

$$\psi^l(0, 0) = H^l(0) = 0$$

and $\psi^l(0, 1) = -\infty$.

For $t = 1, \ldots, l$

$$\psi^l(t, 0) = H^l(t-1)$$

$$\psi^l(t, 1) = \max\{\psi^l(t-1, 1) + y_t^* - d_{tl}z_t^*, \psi^l(t-1, 0) + y_t^* - d_{tl}x_t^*\}$$

$$H^l(t) = \max\{\psi^l(t, 0), \psi^l(t, 1)\}.$$

**Proposition 2.** a. *If* $H^l(l) \leq s_l^*$ *for* $l = 1, \ldots, T$, *none of the inequalities* (2) *is violated by* ($y^*$, $s^*$, $x^*$, $z^*$).
b. *If* $H^l(l) > s_l^*$, *the corresponding inequality is the most violated inequality for that value of* $l$.

**Proof.** The validity of the recursion and the claim follow by observing that

$$\psi^l(t, 1) = \max_{\substack{S\subseteq\{1,\ldots,t\}\\ t\in S}}\left[\sum_{j\in S} y_j^* - \sum_{k=1}^{r}\left(d_{\sigma_k l}x_{\sigma_k}^* + \sum_{i=\sigma_k+1}^{\tau_k} d_{il}z_i^*\right)\right]$$

and

$$\psi^l(t, 0) = \max_{\substack{S\subseteq\{1,\ldots,t\}\\ t\notin S}}\left[\sum_{j\in S} y_j^* - \sum_{k=1}^{r}\left(d_{\sigma_k l}x_{\sigma_k}^* + \sum_{i=\sigma_k+1}^{\tau_k} d_{il}z_i^*\right)\right].$$

In the next section we will see how the dynamic programming recursion also can be written as a shortest path problem.

In the computational experiments of Section 5 we did not use the above separation algorithm. Instead we experimented using a subset of $O(KT)$ inequalities of the form (2) with $r = 1$, $S_1 = S$, $S = \{l - k + 1, \ldots, l\}$ and $k \leq K$. Results are reported with $K \leq 3$ and $K \leq 5$. We refer to the resulting model as **SIM**.

## 2. A POLYNOMIAL REFORMULATION

In this and the next section, we study reformulations that have a polynomial number of constraints and variables in place of the $O(2^T)$ inequalities of the form (2) or (3). Let

$$Y = \{(y, x, z) \in R_+^{3T}: (y, x, z) \text{ satisfy the inequalities (3) and } z \leq x\}.$$

Consider the polyhedron $Z$, where $\pi_{il}$ can be interpreted as the amount produced in $i$ for the periods $i$ through $l$

$$\pi_{il} \leq y_i \qquad \text{for all } i, l \quad \text{with } i \leq l$$

$$\pi_{il} \leq d_{il}x_i \qquad \text{for all } i, l \quad \text{with } i \leq l$$

$$\sum_{k=i}^{t} \pi_{kl} \leq d_{il}x_i + \sum_{k=i+1}^{t} d_{kl}z_k \qquad \text{for all } i, t, l \quad \text{with } i < t \leq l \qquad (4)$$

$$\sum_{i=1}^{l} \pi_{il} \geq d_{1l} \qquad \text{for all } l$$

$$z \leq x, \quad y, x, z \geq 0.$$

We show that the polyhedron $Y$ can be represented by the polyhedron $Z$ described in (4), using an observation of Martin (1987).

**Proposition 3.** *Proj*$_{y,x,z}Z = Y$.

**Proof.** Given a point ($y^*$, $x^*$, $z^*$) that satisfies $z^* \leq x^*$, the separation problem for the inequalities (3) can be formulated as the integer program

$$\min \zeta,\ \zeta = \sum_{l=1}^{T}\left[\sum_{i=1}^{l}\{y_i^*\alpha_{il} + d_{il}x_i^*\beta_{il} + d_{il}z_i^*\gamma_{il}\} - d_{1l}\delta_l\right]$$

$$\begin{aligned}
&\alpha_{il} + \beta_{il} + \gamma_{il} - \delta_l = 0 && \text{for all } i, l \quad \text{with } i \leqslant l \\
&\beta_{il} + \gamma_{il} - \gamma_{i+1,l} \geqslant 0 && \text{for all } i, l \quad \text{with } i < l \qquad (5)\\
&\sum_l \delta_l = 1 && \\
&\gamma_{1l} = 0 && \text{for all } l \\
&\alpha, \beta, \gamma, \delta \geqslant 0 && \text{and integer}
\end{aligned}$$

with violation occurring if and only if $\zeta < 0$. Observe that because $d_{il}x_i^* \geqslant d_{il}z_i^*$ and because of the form of the objective function, $\beta_{il} = 1$ implies $\gamma_{il} = 0$ and $\alpha_{i-1,l} = 1$. Hence, there exists an optimal solution with $t_{il} = \alpha_{il} - \beta_{i+1,l} \geqslant 0$. Substituting for $t_{il}$ and $s_{il} = \beta_{il} + \gamma_{il} - \gamma_{i+1,l} \geqslant 0$, we obtain for each $l$, the shortest path representation

$$\begin{aligned}
&-\alpha_{1l} - \beta_{1l} && = -\delta_l \\
&\beta_{il} + \gamma_{il} - \gamma_{i+1,l} - s_{i,l} && = 0 \\
& && \text{for } i = 1, \ldots, l \\
&\alpha_{il} - \beta_{i+1,l} \qquad - t_{il} && = 0 \\
& && \text{for } i = 1, \ldots, l \\
&-\alpha_{i+1,l} \qquad + s_{il} + t_{il} && = 0 \\
& && \text{for } i = 1, \ldots, l-1 \\
&s_{ll} + t_{ll} && = \delta_l \\
&\alpha, \beta, \gamma, s, t \geqslant 0 &&
\end{aligned}$$

showing that the linear programming relaxation of (5) always has an optimal integer solution. See Figure 1 for the case $l = 3$.

But we know that a linear program: $\min\zeta$, $\zeta = cx$, $Ax \geqslant b$, $x \geqslant 0$ has an optimal value $\zeta \geqslant 0$ if and only if there exists a $u \geqslant 0$ with $uA \leqslant c$, $ub \geqslant 0$. Taking $(\pi_{il}, \mu_{il}, \nu)$ as the dual variables associated with (5), we obtain that $\zeta \geqslant 0$ if and only if

$$\begin{aligned}
&\pi_{il} && \leqslant y_i^* \\
& && \text{for all } i, l \text{ with } i \leqslant l \\
&\pi_{il} + \mu_{il} && \leqslant d_{il}x_i^* \\
& && \text{for all } i, l \text{ with } i \leqslant l \\
&\pi_{il} + \mu_{il} - \mu_{i-1,l} && \leqslant d_{il}z_i^* \qquad (6)\\
& && \text{for all } i, l \text{ with } 1 < i \leqslant l \\
&-\sum_{i=1}^{l}\pi_{il} + \nu && \leqslant -d_{1l} \\
& && \text{for all } l \\
&\mu \geqslant 0, \quad \nu \geqslant 0 &&
\end{aligned}$$

is feasible.

Finally, we observe that for fixed $l$

$$\begin{aligned}
&\operatorname{proj}_\pi\{(\pi, \mu)\colon \pi_{il} + \mu_{il} \leqslant d_{il}x_i^* \text{ for all } i \leqslant l \\
&\qquad \pi_{il} + \mu_{il} - \mu_{i-1,l} \leqslant d_{il}z_i^* \quad \text{for } 1 < i \leqslant l,\ \mu \geqslant 0\} \\
&= \{\pi\colon \pi_{il} \leqslant d_{il}x_i^* \text{ for all } i \leqslant l \\
&\qquad \sum_{k=i}^{t}\pi_{kl} \leqslant d_{il}x_i^* + \sum_{k=i+1}^{t} d_{kl}z_k^* \\
&\qquad\qquad \text{for all } i, t \text{ with } i < t \leqslant l\}.
\end{aligned}$$

The claim follows.

## 3. TWO MORE REFORMULATIONS

As for the lot-sizing problem **ULS**, it is natural to consider two other formulations based on an

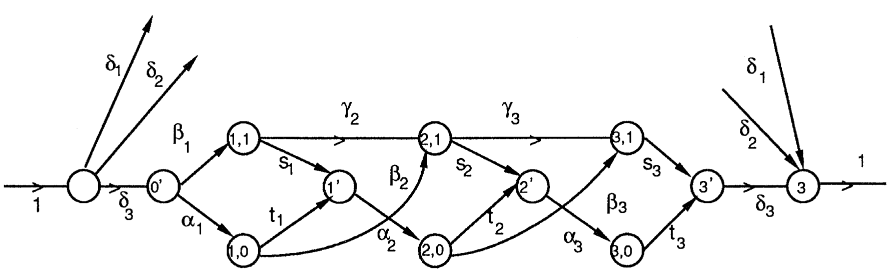


**Figure 1.** Shortest path separation problem.

uncapacitated facility location model and a shortest path model. See Eppen and Martin (1987) and Pochet and Wolsey (1988) for detailed results on problem **ULS**.

Consider first the polyhedron $W^{\text{UFL}}$

$$\sum_{i=1}^{l} y_{il} = d_l \qquad \text{for all } l$$

$$\sum_{t=i}^{k} y_{tl} \leqslant d_l\left(x_i + \sum_{t=i+1}^{k} z_t\right) \qquad \text{for } i, k, l \quad \text{with } i \leqslant k \leqslant l \quad (7)$$

$$z_i \leqslant x_i \qquad \text{for all } i$$

$$\sum_{t=i}^{T} y_{it} = y_i \qquad \text{for all } i$$

$$y_{it} \geqslant 0 \quad \text{for all } i, t \quad \text{with } i \leqslant t, x_i, z_i \geqslant 0 \quad \text{for all } i.$$

Observe that we can interpret $y_{il}$ as the production in period $i$ to satisfy demand in period $l$, so that $\pi_{il} = \sum_{t=i}^{l} y_{it} \leqslant y_i$. We see immediately that the constraints of $Z$ described in (4) are an aggregation of the constraints of $W^{\text{UFL}}$.

**Proposition 4.** *$Proj_{y,x,z} W_{\text{UFL}} \subseteq proj_{y,x,z} Z$*

By substituting $s_i = \sum_{t=1}^{i} x_t - d_{1i}$, the storage costs $h_i$ can be taken as zero in formulation (1). Using $y_i = \sum_{t=i}^{T} y_{it}$ to substitute out the production variables, we obtain the formulation

$$\min\left\{\sum_{i,l} p_i y_{il} + \sum_i c_i x_i + \sum_i f_i z_i: \quad 1 - x_{i-1} \geqslant z_i \geqslant x_i - x_{i-1} \quad \text{for all } i, (y, x, z) \in W^{\text{UFL}}, x, z \in B^T\right\}.$$

Next, assuming for simplicity that $d_i > 0$ for all $i$, it is known from the structure of the optimal solutions of (1) that there exists an optimal solution in which

$$x_i \geqslant \frac{y_{ii}}{d_i} \geqslant \frac{y_{i,i+1}}{d_{i+1}} \geqslant \ldots \geqslant \frac{y_{iT}}{d_T}. \qquad (8)$$

Adding the constraints of (8) to (7) we see that the $(i, k, l)$ constraints of (7) with $k < l$ are dominated by the constraint $(i, k, k)$. Hence, by setting $w_{il} = y_{il}/d_l$ we obtain the equivalent model

$$\min \sum_i \sum_l p_i d_l w_{il} + \sum_i c_i x_i + \sum_i f_i z_i$$

$$\sum_{i=1}^{l} w_{il} = 1 \quad \text{for all } l$$

$$\sum_{t=i}^{k} w_{tk} \leqslant x_i + \sum_{t=i+1}^{k} z_t \quad \text{for all } i, k \text{ with } 1 \leqslant i \leqslant k \qquad (9)$$

$$x_i \geqslant w_{ii} \geqslant w_{i,i+1} \geqslant \ldots \geqslant w_{i,T} \quad \text{for all } i$$

$$\min\{x_i, 1 - x_{i-1}\} \geqslant z_i \geqslant x_i - x_{i-1} \quad \text{for all } i$$

$$w \geqslant 0, \quad x, z \in \{0, 1\}.$$

Finally, introducing the variable $\phi_{il} = w_{il} - w_{i,l+1} \geqslant 0$ in model (9) leads to the shortest path reformulation

$$\min \sum_i \sum_l p_i d_{il} \phi_{il} + \sum_i c_i x_i + \sum_i f_i z_i$$

$$\sum_{t=1}^{T} \phi_{1t} = 1$$

$$\sum_{i=1}^{t-1} \phi_{i,t-1} - \sum_{k=t}^{T} \phi_{tk} = 0 \quad t = 2, \ldots, T \qquad (10)$$

$$\sum_{t=i}^{k} \sum_{s=k}^{T} \phi_{ts} \leqslant x_i + \sum_{t=i+1}^{k} z_t \quad \text{for all } i, k \quad \text{with } 1 \leqslant i \leqslant k$$

$$\min\{x_i, 1 - x_{i-1}\} \geqslant z_i \geqslant x_i - x_{i-1} \quad \text{for all } i$$

$$\phi \geqslant 0, \quad x, z \in \{0, 1\}.$$

We call this a shortest path **SP** model, because a solution to the equality constraints defines a path from 1 to $T$.

At this stage, we know that the linear programming relaxation based on **SP**(10) is at least as strong as that based on $W^{\text{UFL}}$ (7), which is at least as strong as that based on (4) or (6), which is equivalent to **SIM** (consisting of the formulation (1) plus the family of inequalities (2)).

## 4. COMPUTATIONAL RESULTS

We briefly tested three of the models developed in Sections 2–4. The first is the model **SIM** in the space of the original variables $(y, s, x, z)$. Rather than use a special purpose cutting plane algorithm based on Proposition 2, we preferred to add a subset of the inequalities (2) a priori. Thus, anyone with access to a mathematical programming system can repeat these results. For a choice of parameter $K$, we added inequalities with $r = 1$, and $S_1 = S = \{l - k + 1, \ldots, l\}$ for all $l$, $k$ with $k \leq K$. The resulting model has at most $4T$ variables and $O(KT)$ constraints.

The second model tested is a relaxation of the model based on (7) and the change of variable $w_{il} = y_{il}/d_l$, called **UFL**. Here again, for a choice of parameter $K$, we only added the inequalities $\sum_{l=1}^{k} w_{lk} \leq x_i + \sum_{l=i+1}^{k} z_l$ for values of $(i, k)$ that satisfy $k - i \leq K$. The resulting problem has at most $\frac{1}{2}(T^2 + 5T)$ variables and $O(KT)$ constraints.

The **UFL** model is chosen because it is the same size as the **SP** model but it is considerably less dense. It is important to observe that this relaxation is, possibly, a relaxation of the original problem, as the constraints $w_{tl} \leq x_t$ for $l > t + K$ are needed to give a correct formulation.

The third model tested is the model **SP**(10). Here we have only added the inequalities $\sum_{t=i}^{k} \sum_{s=k}^{T} \phi_{ts} \leq x_i + \sum_{l=i+1}^{k} z_l$ for values of $(i, k)$ with $k - i \leq K$. This model has the same number of constraints and variables as the model **UFL**. However, in this case, when $K = T - 1$ we obtain the complete model. The motivation for the parameter $K$ is the hope that if setups and start-ups appear every $K$ periods or less, the corresponding inequalities may suffice.

The data are randomly generated. Two sets of five problems were generated. A 12-period and a 24-period set are obtained using data with the cost $p_t$, $h_t$, $c_t$, $f_t$ integers uniformly distributed in [3, 5], [1, 2], [150, 300], [75, 125], respectively, and the $d_t$ integer uniformly distributed in [50, 100].

Table I shows the results for each of the three models with $K = 3$ for the 12-period problems and $K = 5$ for the 24-period problems. CONV denotes the CONVERT + SETUP time and LP is the Primal Simplex time.

These values of $K$ are, in fact, very conservative for the class of instances generated. In Table II we show the behavior of each of the models with $K = 0$, 1, 2 for one of the 24-period problems. The branch-and-bound times include the linear programming times. The last column indicates whether the solution found was feasible for the original problem. The results indicate that at least for the class of single item problems generated, the strongest and most compact formulation **SP** is not the fastest. The slowness of **SP** is undoubtedly due to the density of the $(i, k)$ inequalities in (10). On the other hand, the complete **SP** formulation requires $O(T^2)$ constraints, the complete **UFL** formulation $O(T^3)$ constraints and **SIM** requires $O(2^T)$ constraints, so one might expect that on more difficult problems this tendency will be reversed. Such problems can be expected to arise when several items are produced on the same machine, so these results also suggest that not just the inequalities (2) but even the $(i, k)$ inequalities in (10) and the $(i, k, l)$ inequalities in (9) should be generated as cuts rather than added a priori.

Given the results, it is also natural to conjecture that the different models suffice to describe the convex hull of solutions to the original problem (1), as is the case without the start-up variables $z$ (see Barany, Van Roy and Wolsey 1984b). The reformulation of Eppen and Martin based on a dynamic programming recursive is known to describe the convex hull of the solutions. It contains $O(T^2)$ variables and $O(T^2)$ constraints like the **SP** formulation.

We also solved some multi-item uncapacitated problems with changeover costs. This model is

**Table I**
The Single Item Start-Up Model

| | $T$ | $K$ | ROWS | COLS | DENS | secs CONV | secs LP | PIVS |
|---|---|---|---|---|---|---|---|---|
| SIM | 12 | 3 | 90 | 47 | 84 | 1.68 | 2.4 | 77 |
| | | | | | | | 2.2–3.1 | 70–97 |
| UFL | 12 | 3 | 77 | 101 | 4.2 | 2.1 | 2.5 | 103 |
| | | | | | | | 2.2–2.7 | 90–111 |
| SP | 12 | 3 | 77 | 101 | 10.8 | 3.4 | 3.7 | 116 |
| | | | | | | | 3.0–4.3 | 93–132 |
| SIM | 24 | 5 | 225 | 95 | 5.7 | 5.3 | 13.2 | 211 |
| | | | | | | | 10.5–16.3 | 175–245 |
| UFL | 24 | 5 | 200 | 347 | 1.8 | 6.3 | 15.0 | 288 |
| | | | | | | | 14.1–16.1 | 276–309 |
| SP | 24 | 5 | 200 | 347 | 8.4 | 18.7 | 32.7 | 320 |
| | | | | | | | 27.3–35.5 | 276–351 |

**Table II**
Adding Selected Inequalities a Priori

| | $K$ | ROWS | COLS | Linear Program CONV | LP | PIVS | Branch-and-Bound NODES | SECS | PIVS | Values LP | IP | FEAS. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| SIM | 0 | 120 | 95 | 2.4 | 3.5 | 112 | 400+ | 140+ | 745+ | 10677 | 11938[a] | Y |
| SIM | 1 | 143 | 95 | 2.8 | 4.4 | 133 | 3 | 6.1 | 137 | 11792 | 11847 | Y |
| SIM | 2 | 165 | 95 | 3.3 | 5.3 | 137 | | | | 11847 | | Y |
| UFL | 1 | 118 | 347 | 3.9 | 4.4 | 112 | | | | 11787 | | N |
| UFL | 2 | 140 | 347 | 4.4 | 6.8 | 164 | | | | 11847 | | Y |
| SP | 0 | 95 | 347 | 5.1 | 10.8 | 226 | 23 | 21.7 | 257 | 11640 | 11847 | Y |
| SP | 1 | 118 | 347 | 6.7 | 12.0 | 219 | | | | 11847 | | Y |
| SP | 2 | 140 | 347 | 9.4 | 15.6 | 246 | | | | 11847 | | Y |

[a] Indicates the value of the best solution found after 400 nodes.

presented but not tackled in Karmarkar and Schrage. The basic formulation is

$$\min \sum_{i,t} p_{it}y_{it} + \sum_{i,t} h_{it}s_{it} + \sum_{i,t} c_{it}x_{it} + \sum_{i,j,t} q_{ij}v_{ijt}$$

$$s_{i,t-1} + y_{it} = d_{it} + s_{it} \quad \text{for all } i, t$$

$$y_{it} \leq Mx_{it} \quad \text{for all } i, t$$

$$\sum_i x_{it} = 1 \quad \text{for all } t$$

$$v_{ijt} \geq x_{i,t-1} + x_{jt} - 1 \quad \text{for all } i, j, t$$

$$s, y \geq 0, \quad x, v \in \{0, 1\}$$

where $y_{it}$, $s_{it}$, $x_{it}$, denote the production, storage and setup of item $i$ in period $t$, and $v_{ijt} = 1$ if item $i$ is setup in period $t - 1$ and item $j$ in period $t$. Note that in this model only one item can be produced in each period.

This formulation is strengthened by dropping the last set of constraints that define $v_{ijt}$, and replacing them by

$$z_{jt} = \sum_{\{i:i \neq j\}} v_{ijt}$$

$$x_{i,t-1} = \sum_j v_{ijt}$$

$$x_{it} = z_{it} + v_{iit}$$

where $z_{it}$ is the start-up variable for item $i$ in period $t$. We can use any of the reformulations for the start-up variables described earlier.

Based on the single item results we choose to use reformulation **UFL** plus the additional constraints $\sum_s w_{its} \leq Tx_{it}$, where $w_{its}$ is the fraction of the demand for $i$ in period $s$ produced in period $t$. These additional constraints serve two purposes. First, they guarantee a valid formulation, and second, the inequalities $w_{its} \leq x_{it}$ are generated as cuts from these constraints if they are violated.

In Table III we indicate the results for four problems—CUTS indicates the number of constraints of the form $w_{its} \leq x_{it}$ that are added. XLP denotes the total time to solve the linear program, generate cuts and reoptimize until no more cuts are found. Surprisingly, an integer solution is obtained for all four problems without using branch-and-bound. The data are randomly generated with $p_{it} = q_{ii} = 0$, $h_{it}$, $c_{it}$, $q_{ij}(i \neq j)$ and $d_{it}$ uniformly distributed and integer in [1, 2], [100, 400], [200, 900] and [50, 100], respectively, and initial and final stocks in [200, 350].

The results for single-item problems were obtained using SCICONIC Version 1.32 on Data General MV8000, and the results for multi-item problems used SCICONIC Version 1.20 + MPSARX, to generate the violated inequalities $w_{its} \leq x_{it}$ (see Van Roy and Wolsey 1987).

**Table III**
Multi-Item Model With Sequence-Dependent Costs

| | $I$ | $T$ | $K$ | ROWS | COLS | 0–1 | DENS | secs CONV | secs LP | PIVS | CUTS | secs XLP | PIVS |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| TSP01 | 5 | 10 | 6 | 503 | 601 | 275 | .920 | 15 | 206 | 980 | 21 | 223 | 1010 |
| TSP02 | 5 | 12 | 6 | 656 | 786 | 335 | .784 | 21 | 290 | 1103 | 31 | 344 | 1232 |
| TSP03 | 5 | 15 | 6 | 848 | 1101 | 425 | .596 | 29 | 794 | 2147 | 37 | 994 | 2579 |
| TSP04 | 5 | 15 | 6 | 848 | 1101 | 425 | .596 | 30 | 736 | 1967 | 44 | 874 | 2238 |

Since being accepted for publication, further progress has been made on the problem treated here. S. Van Hoesel discovered a fractional solution that is not cut off by the inequalities (2). We then showed that the slightly more general inequality family (2′) in which the term $d_{\sigma_k l} x_{\sigma_k}$ can be replaced by the term: $d_{\sigma_k l}(z_{\tau_{k-1}+1} + \ldots + z_{\sigma_k})$ for $k > 1$ is valid for $X$ and cuts off the point he proposed. He has shown that this new family describes conv($X$) when added to the initial formulation. This result implies that that $\text{proj}_{y,x,z} Z \supset \text{conv } X$, whereas $\text{proj}_{y,x,z} W^{\text{UFL}} = \text{conv } X$, and thus the formulation $W^{\text{UFL}}$ is tight.

## ACKNOWLEDGMENT

This research was supported by the Projet d'Action Concertée No. 87/92-106 of CORE.

## REFERENCES

BARANY, I., T. J. VAN ROY AND L. A. WOLSEY. 1984a. Strong Formulations for Multi-Item Capacitated Lot-Sizing. *Mgmt. Sci.* **30,** 1255–1261.

BARANY, I., T. J. VAN ROY AND L. A. WOLSEY. 1984b. Uncapacitated Lot-Sizing: The Convex Hull of Solutions. *Math. Prog. Study* **22,** 32–43.

EPPEN, G. D., AND R. K. MARTIN. 1987. Solving Multi-Item Capacitated Lot-Sizing Problems Using Variable Redefinition. *Opns. Res.* **35,** 832–848.

FLEISCHMANN, B. 1987. The Discrete Lot-Sizing and Scheduling Problem. Institüt für Unternehmungsforschung, Universität, Hamburg.

KARMARKAR, U. S., AND L. SCHRAGE. 1985. The Deterministic Dynamic Cycling Problem. *Opns. Res.* **33,** 326–345.

MARTIN, R. K. 1987. Using Separation Algorithms to Generate Mixed Integer Model Reformulations. Graduate School of Business, University of Chicago, Chicago.

POCHET, Y., AND L. A. WOLSEY. 1988. Lot-Sizing Models With Backlogging: Strong Reformulations and Cutting Planes. *Math. Prog.* **40,** 317–335.

VAN ROY, T. J., AND L. A. WOLSEY. 1987. Solving Mixed Integer Programming Problems Using Automatic Reformulation. *Opns. Res.* **35,** 45–57.

VAN WASSENHOVE, L. N., AND P. VANDERHEUST. 1983. Planning Production in a Bottleneck Department: A Case Study. *Eur. J. Opns. Res.* **12,** 127–137.