11. INTEGER VARIABLES.

In applications, one is often led to seek <u>integers</u> $x_1, x_2, \ldots, x_n$ which maximize a linear function subject to linear constraints. For example, consider a hypothetical round-the-clock telephone switchboard where operators work on nine-hour shifts. Regulations require that each operator works on the same shift every day (or night). There are exactly eight shifts over the 24-hour period:

the first shift	...	midnight	till	9 AM
the second shift	...	9 AM	till	noon
the third shift	...	6 AM	till	3 PM
the fourth shift	...	9 AM	till	6 PM
the fifth shift	...	noon	till	9 PM
the sixth shift	...	3 PM	till	midnight
the seventh shift	...	6 PM	till	3 AM
the eighth shift	...	9 PM	till	6 AM

The number of operators required at the switchboard varies with the time of day as indicated below:

midnight	till	6 AM	...	one operator
6 AM	till	9 AM	...	two operators
9 AM	till	noon	...	five operators
noon	till	6 PM	...	six operators
6 PM	till	9 PM	...	three operators
9 PM	till	midnight	...	two operators.

The task, of course, is to satisfy these requirements by as few operators as possible. Denoting by x_j the number of operators working on the j-th shift, we are led to

$$\left.\begin{array}{ll}
\text{minimize} & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \\
\text{subject to} & x_1 + x_7 + x_8 \geq 1 \\
& x_1 + x_2 + x_8 \geq 1 \\
& x_1 + x_2 + x_3 \geq 2 \\
& x_2 + x_3 + x_4 \geq 5 \\
& x_3 + x_4 + x_5 \geq 6 \\
& x_4 + x_5 + x_6 \geq 6 \\
& x_5 + x_6 + x_7 \geq 3 \\
& x_6 + x_7 + x_8 \geq 2 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0 \ .
\end{array}\right\} \quad (1)$$

The optimal solution of this LP problem is

$$x_1 = x_2 = \frac{1}{3} \ , \quad x_3 = \frac{4}{3} \ , \quad x_4 = \frac{10}{3} \ , \quad x_5 = x_6 = \frac{4}{3} \ , \quad x_7 = x_8 = \frac{1}{3} \ . \qquad (2)$$

For our application, these numbers are useless: we have forgotten to impose the natural requirement that

$$\text{each } x_j = \text{integer} \qquad (j = 1, 2, \ldots, 8) \ .$$

Adding this "integrality constraint" to the constraints of (1) we obtain an <u>integer linear programming problem</u> (or an ILP problem for short); the optimal solution (2) of (1) is no longer feasible in the ILP problem. There are various ways of solving such problems; in this section, we shall confine ourselves to an elegant method developed by R. E. Gomory.

To begin with, note that the sum of the eight linear constrains in (1) reads

$$3 \sum_{j=1}^{8} x_j \geq 26$$

or, when divided by three,

$$\sum_{j=1}^{8} x_j \geq \frac{26}{3} .$$

Hence every satisfactory schedule has to use at least 26/3 operators. On the other hand, the number of operators used in any schedule is an integer. An integer not less than 26/3 is necessarily at least nine; we conclude that a satisfactory schedule has to use at least nine operators. Note that the inequality

$$\sum_{j=1}^{8} x_j \geq 9 , \tag{3}$$

although implied by the linear constraints of (1) together with the integrality constraint, is not implied by the constraints of (1) alone: indeed, (2) satisfies the linear constraints of (1) but violates (3). Such an inequality is called a cutting plane or simply a cut. The concept of a cut is crucial for Gomory's algorithm.

To illustrate the algorithm, we shall

$$\left.\begin{array}{lrcl} \text{maximize} & 2x_1 + 3x_2 + 2x_3 & & \\ \text{subject to} & x_1 + x_2 + x_3 & \leq & 8 \\ & 2x_1 - x_2 \qquad & \leq & 3 \\ & 3x_2 + x_3 & \leq & 10 \\ & x_1, x_2, x_3 & \geq & 0 \\ & x_1, x_2, x_3 & = & \text{integer.} \end{array}\right\} \tag{4}$$

In order to solve (4), we shall first consider the ordinary LP problem obtained from (4) by dropping the integrality constraint:

$$
\left.\begin{array}{lrcl}
\text{maximize} & 2x_1 + 3x_2 + 2x_3 & & \\
\text{subject to} & x_1 + x_2 + x_3 & \leq & 8 \\
& 2x_1 - x_2 & \leq & 3 \\
& 3x_2 + x_3 & \leq & 10 \\
& x_1, x_2, x_3 & \geq & 0 \quad .
\end{array}\right\} \qquad (5)
$$

It is not very likely that an optimal solution of (5) consists of <u>integers</u> x_1 , x_2 , x_3 ; however, if such a solution does exist then it automatically constitutes an optimal solution of (4). Inspired by this observation, we shall proceed to solve (5). Applying the simplex method, we begin with the initial table

$$
\left.\begin{array}{rcl}
x_4 & = & 8 - x_1 - x_2 - x_3 \\
x_5 & = & 3 - 2x_1 + x_2 \\
x_6 & = & 10 - 3x_2 - x_3 \\
\hline
z & = & 2x_1 + 3x_2 + 2x_3
\end{array}\right\} \qquad (6)
$$

and, after three iterations, arrive at the final table

$$
\left.\begin{array}{rcl}
x_1 & = & \frac{8}{3} + \frac{1}{3} x_4 - \frac{2}{3} x_5 - \frac{1}{3} x_6 \\
x_2 & = & \frac{7}{3} + \frac{2}{3} x_4 - \frac{1}{3} x_5 - \frac{2}{3} x_6 \\
x_3 & = & 3 - 2 x_4 + x_5 + x_6 \\
\hline
z & = & \frac{55}{3} - \frac{4}{3} x_4 - \frac{1}{3} x_5 - \frac{2}{3} x_6 \quad .
\end{array}\right\} \qquad (7)
$$

The resulting optimal solution (8/3 , 7/3 , 3) fails to satisfy the integrality constraint of (4). Nevertheless, the final able (7) contains valuable information which will bring us closer to solving (4). More precisely, it points out a cut.

The last row of (7) tells us that every feasible solution of (5) satisfies the inequality $z \leq 55/3$. Therefore every feasible solution of (4), being a feasible solution of (5), satisfies the same inequality. In addition, every feasible solution of (4) gives an <u>integer</u> value to z . That integer, not exceeding the non-integer $55/3$, will certainly not exceed 18 . In short, every feasible solution of (4) satisfies the inequality

$$z = 2x_1 + 3x_2 + 2x_3 \leq 18 . \tag{8}$$

Note that (8) is <u>not</u> satisfied by every feasible solution of (5): indeed, $(8/3, 7/3, 3)$ violates (8). Hence (8) is a cut.

A more careful analysis of the last row of (7) will produce a stronger cut than (8). Indeed, that row may be written as

$$(z + x_4) + \left(\frac{1}{3} x_4 + \frac{1}{3} x_5 + \frac{2}{3} x_6 \right) = \frac{55}{3} . \tag{9}$$

For every feasible solution (x_1, x_2, x_3) of (4), and for the corresponding values of x_4, x_5, x_6, z defined by (6), the first bracket on the left-hand side of (9) is an integer and the second bracket is nonnegative. Thus for every feasible solution of (4), the quantity $z + x_4$ is an integer not exceeding $55/3$; we conclude that every feasible solution of (4) satisfies the inequality

$z \leq 18$

$$z + x_4 \leq 18 . \tag{10}$$

Clearly, (10) is a stronger cut than (8). Substituting for z and x_4 from (6), we may express (10) in terms of the original variables as

$$x_1 + 2x_2 + x_3 \leq 10 . \tag{10'}$$

Now we are led to consider a new LP problem obtained by adding the cut (10') to the constraints of (5):

$$\left.\begin{array}{lrcl} \text{maximize} & 2x_1 + 3x_2 + 2x_3 & & \\ \text{subject to} & x_1 + x_2 + x_3 & \leq & 8 \\ & 2x_1 - x_2 & \leq & 10 \\ & 3x_2 + x_3 & \leq & 10 \\ & x_1 + 2x_2 + x_3 & \leq & 10 \\ & x_1, x_2, x_3 & \geq & 0 \quad . \end{array}\right\} \qquad (11)$$

This problem and (5) are related to (4) in a similar way. Indeed, every feasible solution of (4) is a feasible solution of (11); conversely, every <u>integer</u> feasible solution of (11) is a feasible solution of (4). In addition, the two problems (4) and (11) have the same objective function. Linear programming problems with these properties are called <u>relaxations</u> of (4). As before, we shall proceed to solve (11) with the aim of obtaining either an optimal solution of (4) or a new cut.

It would be silly and inefficient to solve (11) from scratch. Indeed, this problem has been obtained by adding a new constraint to the problem (5) that we have already solved. In such a situation, the dual simplex method is called for: we shall add to (7) a formula expressing the new slack variable in terms of the nonbasic variables x_4, x_5, x_6 and then we shall pivot to restore primal feasibility. From (10'), we may express the new slack variable as

$$x_7 = 10 - x_1 - 2x_2 - x_3 \quad ;$$

substituting for x_1, x_2 and x_3 from (6), we may convert this formula into the desired form. However, there is an easy direct way of expressing

x_7 in terms of x_4, x_5 and x_6. Indeed, from the identity (9) we see that our cut (10) is equivalent to

$$\frac{1}{3}x_4 + \frac{1}{3}x_5 + \frac{2}{3}x_6 \geq \frac{1}{3}$$

and so

$$x_7 = -\frac{1}{3} + \frac{1}{3}x_4 + \frac{1}{3}x_5 + \frac{2}{3}x_6 .$$

Hence the new table reads

$$\begin{array}{lcl}
x_1 & = & \frac{8}{3} + \frac{1}{3}x_4 - \frac{2}{3}x_5 - \frac{1}{3}x_6 \\
x_2 & = & \frac{7}{3} + \frac{2}{3}x_4 - \frac{1}{3}x_5 - \frac{2}{3}x_6 \\
x_3 & = & 3 - 2x_4 + x_5 + x_6 \\
x_7 & = & -\frac{1}{3} + \frac{1}{3}x_4 + \frac{1}{3}x_5 + \frac{2}{3}x_6 \\
\hline
z & = & \frac{55}{3} - \frac{4}{3}x_4 - \frac{1}{3}x_5 - \frac{2}{3}x_6 .
\end{array}$$

Pivoting (with x_7 leaving and x_6 entering) we obtain

$$\left.\begin{array}{lcl}
x_6 & = & \frac{1}{2} - \frac{1}{2}x_4 - \frac{1}{2}x_5 + \frac{3}{2}x_7 \\
x_1 & = & \frac{5}{2} + \frac{1}{2}x_4 - \frac{1}{2}x_5 - \frac{1}{2}x_7 \\
x_2 & = & 2 + x_4 - x_7 \\
x_3 & = & \frac{7}{2} - \frac{5}{2}x_4 + \frac{1}{2}x_5 + \frac{3}{2}x_7 \\
\hline
z & = & 18 - x_4 - x_7 .
\end{array}\right\} \quad (12)$$

This table describes an optimal solution $(5/2, 2, 7/2)$ of (11) which fails to satisfy the integrality constraint of (4). Note that the current value of z is an integer and so the last row of (12) yields no cut. However, nothing prevents us from applying the cut-producing technique to, say, the row that expresses x_1 in (12). That row may be written as

$$(x_1 - x_4) + \left(\frac{1}{2} x_4 + \frac{1}{2} x_5 + \frac{1}{2} x_7 \right) = \frac{5}{2} \quad . \qquad (13)$$

As before, we conclude that every feasible solution of (4) satisfies the inequality

$$x_1 - x_4 \leq 2 \quad ;$$

adding this cut to the constraints of (11) we obtain a new relaxation of (4). We may formulate the new relaxation explicitly as

$$\left.\begin{array}{lrcl}
\text{maximize} & 2x_1 + 3x_2 + 2x_3 & & \\
\text{subject to} & x_1 + x_2 + x_3 & \leq & 8 \\
 & 2x_1 - x_2 & \leq & 3 \\
 & 3x_2 + x_3 & \leq & 10 \\
 & x_1 + 2x_2 + x_3 & \leq & 10 \\
 & 2x_1 + x_2 + x_3 & \leq & 10 \\
 & x_1, x_2, x_3 & \geq & 0 \quad .
\end{array}\right\} \qquad (14)$$

However, it is not necessary to do that. In order to solve (14), we only have to extract from (13) a formula for the new variable x_8 in terms of x_4 , x_5 , x_7 , then add this formula to (12) and apply the dual simplex method. Hence we begin with the table

$$\begin{array}{lcl}
x_6 & = & \frac{1}{2} - \frac{1}{2} x_4 - \frac{1}{2} x_5 + \frac{3}{2} x_7 \\
x_1 & = & \frac{5}{2} + \frac{1}{2} x_4 - \frac{1}{2} x_5 - \frac{1}{2} x_7 \\
x_2 & = & 2 + x_4 - x_7 \\
x_3 & = & \frac{7}{2} - \frac{5}{2} x_4 + \frac{1}{2} x_5 + \frac{3}{2} x_7 \\
x_8 & = & -\frac{1}{2} + \frac{1}{2} x_4 + \frac{1}{2} x_5 + \frac{1}{2} x_7 \\
\hline
z & = & 18 - x_4 - x_7 \quad .
\end{array}$$

Then we pivot, with x_8 leaving and x_5 entering, and obtain

$$\begin{array}{rcl} x_5 & = & 1 - x_4 - x_7 + 2x_8 \\ x_6 & = & 2x_7 - x_8 \\ x_1 & = & 2 + x_4 - x_8 \\ x_2 & = & 2 + x_4 - x_7 \\ x_3 & = & 4 - 3x_4 + x_7 + x_8 \\ \hline z & = & 18 - x_4 - x_7 \end{array}$$

This time, luck is finally on our side: the last table describes an optimal solution (2,2,4) of (14). This solution consists of integers and so it constitutes an optimal solution of (4).

The cut-producing technique illustrated above applies to all the problems with "all integer data". These are the problems

$$\left.\begin{array}{lll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & (i = 1,2,\ldots,m) \\ & x_j \geq 0 & (j = 1,2,\ldots,n) \\ & x_j = \text{integer} & (j = 1,2,\ldots,n) \end{array}\right\} \quad (15)$$

such that all the numbers a_{ij}, b_i and c_j are integers. The "all integer" property guarantees that, for every choice of integers $x_1, x_2, \ldots, x_n$, all the slack variables $x_{n+1}, x_{n+2}, \ldots, x_m$ and the objective function z assume integer values.

In order to provide an overview of Gomory's algorithm, we shall solve another ILP problem with all-integer data:

$$\left.\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & 7x_1 - 3x_2 \leq 7 \\ & -2x_1 + 3x_2 \leq 3 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 = \text{integer.} \end{array}\right\} \quad (16)$$

Dropping the integrality constraint from (16), we obtain its relaxation

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & 7x_1 - 3x_2 \leq 7 \\ & -2x_1 + 3x_2 \leq 3 \\ & x_1, x_2 \geq 0 \; . \end{array}$$

Solving this LP problem by the simplex method, we arrive at the final table

$$\begin{array}{lcl} x_1 & = & 2 - \frac{1}{5}x_3 - \frac{1}{5}x_4 \\ x_2 & = & \frac{7}{3} - \frac{2}{15}x_3 - \frac{7}{15}x_4 \\ \hline z & = & \frac{13}{3} - \frac{1}{3}x_3 - \frac{2}{3}x_4 \; . \end{array}$$

For a moment, let us consider the general situation when we have arrived at the final table for some relaxation of an ILP problem (15) with all-integer data. Denoting by N the set of "nonbasic subscripts" we may write the last row of that table as

$$z = v - \sum_{j \in N} d_j x_j \; . \qquad (17)$$

If v is not an integer then (17) may be used to produce a cut. Indeed, writing $\lfloor t \rfloor$ for the largest integer not exceeding t (so that

$\lfloor 1/2 \rfloor = 0$, $\lfloor -1/3 \rfloor = -1$, etc.) we may write (17) as

$$\left(z + \sum_{j \in N} \lfloor d_j \rfloor x_j \right) + \sum_{j \in N} (d_j - \lfloor d_j \rfloor) x_j = v \quad . \tag{18}$$

If $(x_1, x_2, \ldots, x_n)$ is a feasible solution of (15) then the first bracket has an integer value whereas the second bracket has a nonnegative value. We conclude that every feasible solution of (15) satisfies the inequality

$$z + \sum_{j \in N} \lfloor d_j \rfloor x_j \leq \lfloor v \rfloor \quad ; \tag{19}$$

this inequality is our cut. Adding (19) to the constraints of the previous relaxation, we obtain a new relaxation of (15). In order to solve this relaxation, we first observe that (19) is equivalent to

$$\sum_{j \in N} (d_j - \lfloor d_j \rfloor) x_j \geq v - \lfloor v \rfloor$$

and so its slack variable (say x_{k+1}) can be expressed as

$$x_{k+1} = (\lfloor v \rfloor - v) + \sum_{j \in N} (d_j - \lfloor d_j \rfloor) x_j \quad . \tag{20}$$

Then we add (20) to our last table and apply the dual simplex method to restore primal feasibility. It is important to note that, for every choice of integers $x_1, x_2, \ldots, x_n$, the new variable x_{k+1} assumes an integer value. That may not be obvious from (20); however, it does follow quite easily from the equivalent formula

$$x_{k+1} = \lfloor v \rfloor - z - \sum_{j \in N} \lfloor d_j \rfloor x_j \quad .$$

Returning to our example, we add the formula

$$x_5 = -\frac{1}{3} + \frac{1}{3} x_3 + \frac{2}{3} x_4$$

to our last table and apply the dual simplex method. One iteration (with x_5 leaving and x_3 entering) leads to

$$\begin{array}{lcl} x_1 & = & \frac{9}{5} + \frac{1}{5} x_4 - \frac{3}{5} x_5 \\ x_2 & = & \frac{11}{5} - \frac{1}{5} x_4 - \frac{2}{5} x_5 \\ x_3 & = & 1 - 2 x_4 + 3 x_5 \\ \hline z & = & 4 \qquad\qquad - x_5 \end{array} .$$

That table brings up the general problem of finding a cut when z has an integer value.

In that general case, we may assume that at least one of the original variables $x_1, x_2, \ldots, x_n$ has a non-integer value. (If that is <u>not</u> the case then we are done: since the optimal solution of the last relaxation consists of integers, it constitutes an optimal solution of the ILP problem.) The row expressing that variable,

$$x_i = b - \sum_{j \in N} a_j x_j \quad ,$$

may be written as

$$\left(x_i + \sum_{j \in N} \lfloor a_j \rfloor x_j \right) + \sum_{j \in N} (a_j - \lfloor a_j \rfloor) x_j \leq b$$

and yields the cut

$$x_i + \sum_{j \in N} \lfloor a_j \rfloor x_j \leq \lfloor b \rfloor$$

which is equivalent to

$$\sum_{j \in N} (a_j - \lfloor a_j \rfloor) x_j \geq b - \lfloor b \rfloor \quad .$$

The new slack variable

$$x_{k+1} = (\lfloor b \rfloor - b) + \sum_{j \in N} (a_j - \lfloor a_j \rfloor) x_j$$

assumes integer values for every choice of integers $x_1, x_2, \ldots, x_n$; indeed, we have

$$x_{k+1} = \lfloor b \rfloor - x_i - \sum_{j \in N} \lfloor a_j \rfloor x_j \quad .$$

The whole procedure is no different from that of deriving cuts from the formula for z . In fact, any row with a non-integer absolute term can be used to produce a cut. The row that is actually used is called the <u>source row</u>.

Returning to our example once again, we let the x_1 -row be the source row. Thus we add

$$x_6 = -\frac{4}{5} + \frac{4}{5} x_4 + \frac{3}{5} x_5$$

to our last table and then apply the dual simplex method. After two iterations we obtain the table

$$\begin{array}{lcl}
x_5 & = & \frac{2}{9} + \frac{2}{9} x_3 + \frac{5}{9} x_6 \\
x_4 & = & \frac{5}{6} - \frac{1}{6} x_3 + \frac{5}{6} x_6 \\
x_1 & = & \frac{11}{6} - \frac{1}{6} x_3 - \frac{1}{6} x_6 \\
x_2 & = & \frac{35}{18} - \frac{1}{18} x_3 - \frac{7}{18} x_6 \\
\hline
z & = & \frac{34}{9} - \frac{2}{9} x_3 - \frac{5}{9} x_6 \quad .
\end{array}$$

This table illustrates a minor feature of the algorithm that has not come up so far: note that for every choice of nonnegative x_3 and x_6 , the variable x_5 assumes a positive value. The constraint $x_5 \geq 0$, being implied by $x_5 \geq 0$ and $x_6 \geq 0$, is superfluous and so we may delete the x_5 row from the last table without affecting the problem. More generally, from a final table for some relaxation, we may delete a row

$$x_i = b + \sum_{j \in N} d_j x_j$$

as long as x_i is a "superfluous slack" variable (that is, as long as $i > n$ and $d_j > 0$ for each $j \in N$). Of course, the deletion is purely optional; the only motivation behind it is the desire to decrease the size of the table and to avoid unnecessary calculations in the future. Deleting the x_5 -row from the last table we obtain

$$\begin{array}{lcl} x_4 & = & \frac{5}{6} - \frac{1}{6} x_3 + \frac{5}{6} x_6 \\ x_1 & = & \frac{11}{6} - \frac{1}{6} x_3 - \frac{1}{6} x_6 \\ x_2 & = & \frac{35}{18} - \frac{1}{18} x_3 - \frac{7}{18} x_6 \\ \hline z & = & \frac{34}{9} - \frac{2}{9} x_3 - \frac{5}{9} x_6 \end{array} .$$

Letting the z -row be the source row, we add

$$x_7 = -\frac{7}{9} + \frac{2}{9} x_3 + \frac{5}{9} x_6 .$$

After one iteration of the dual simplex method, we obtain

$$\begin{array}{lcl} x_3 & = & \frac{7}{2} - \frac{5}{2} x_6 + \frac{9}{2} x_7 \\ x_1 & = & \frac{5}{4} + \frac{1}{4} x_6 - \frac{3}{4} x_7 \\ x_2 & = & \frac{7}{4} - \frac{1}{4} x_6 - \frac{1}{4} x_7 \\ x_4 & = & \frac{1}{4} + \frac{5}{4} x_6 - \frac{3}{4} x_7 \\ \hline z & = & 3 \qquad\qquad - x_7 \end{array} .$$

Letting the x_2 -row be the source row, we add

$$x_8 = -\frac{3}{4} + \frac{1}{4} x_6 + \frac{1}{4} x_7 \quad .$$

After two iterations of the dual simplex method, we obtain

$$\begin{array}{lcl}
x_7 & = & \frac{4}{7} + \frac{1}{7} x_3 + \frac{10}{7} x_8 \\
x_6 & = & \frac{17}{7} - \frac{1}{7} x_3 + \frac{18}{7} x_8 \\
x_1 & = & \frac{10}{7} - \frac{1}{7} x_3 - \frac{3}{7} x_8 \\
x_2 & = & 1 \qquad\qquad - x_8 \\
x_4 & = & \frac{20}{7} - \frac{2}{7} x_3 + \frac{15}{7} x_8 \\
\hline
z & = & \frac{17}{7} - \frac{1}{7} x_3 - \frac{10}{7} x_8 \quad .
\end{array}$$

Now the superfluous x_7 -row may be deleted. Then, letting the z -row be the source row, we add

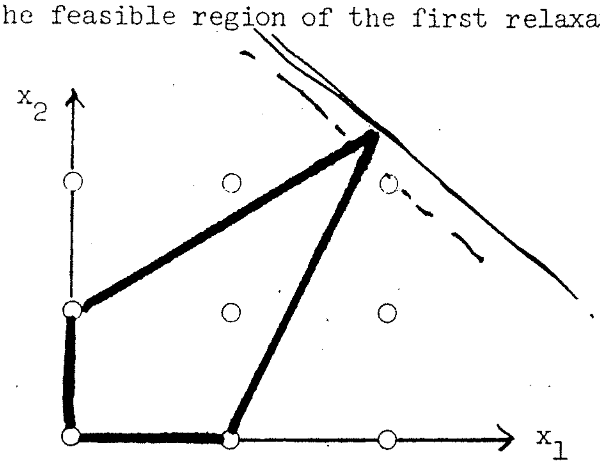
$$x_9 = -\frac{3}{7} + \frac{1}{7} x_3 + \frac{3}{7} x_8 \quad .$$

After one iteration of the dual simplex method, we obtain

$$\begin{array}{lcl}
x_3 & = & 3 - 3x_8 + 7x_9 \\
x_6 & = & 2 + 3x_8 - x_9 \\
x_1 & = & 1 \qquad\quad - x_9 \\
x_2 & = & 1 - x_8 \\
x_4 & = & 2 + 3x_8 - 2x_9 \\
\hline
z & = & 2 - x_8 - x_9 \quad .
\end{array}$$

This table describes an optimal solution (1,1) of (16).

The reader may find it illuminating to review our second example in terms of geometry. The feasible region of the first relaxation of (16) is depicted below.

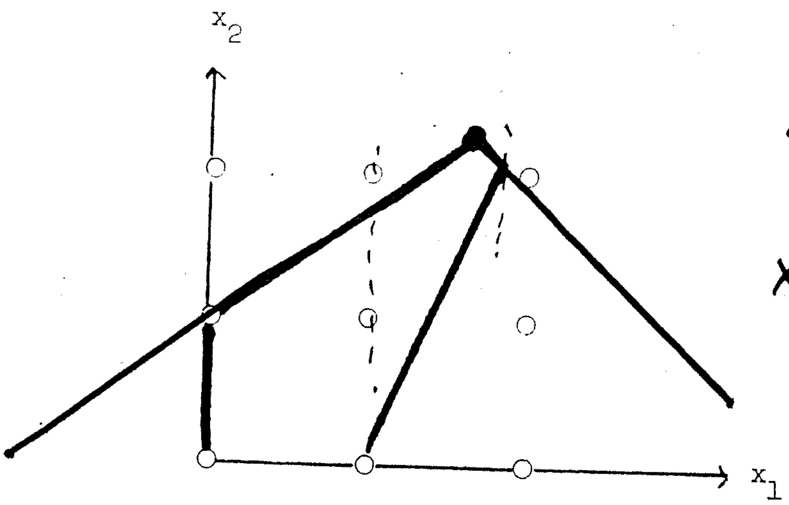


The small circles represent points with integer coordinates x_1 , x_2 ; it is obvious that (16) has only four feasible solutions: (0,0) , (0,1) , (1,0) and (1,1) . Solving (16), we have introduced five cuts in a succession. Here they go, expressed in terms of x_1 and x_2 :

the first cut	...	$x_1 + x_2 \leq 4$
the second cut	...	$-x_1 + 3x_2 \leq 4$
the third cut	...	$x_1 + x_2 \leq 3$
the fourth cut	...	$x_2 \leq 1$
the fifth cut	...	$x_1 \leq 1$.

Introducing each new cut in this sequence, we have obtained a new relaxation of (16); its feasible region resulted by "cutting off" a part of the feasible region of the previous relation. This process is illustrated below.

cut on x_1



After the first cut

$x_1 = \frac{9}{5} + \frac{1}{5}x_4 - \frac{3}{5}x_5$

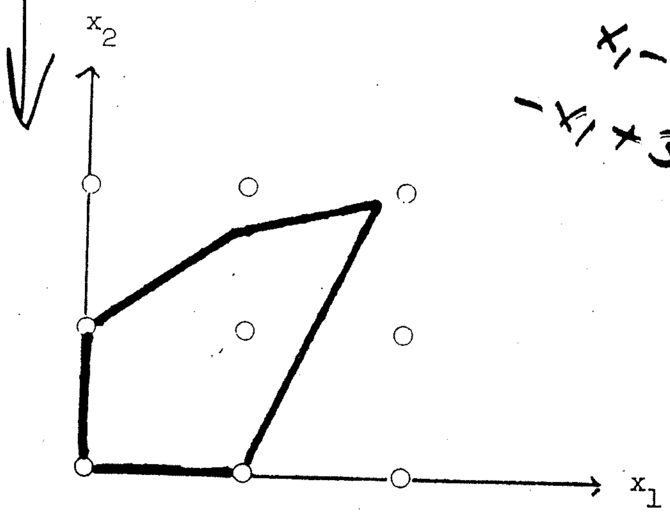
$x_1 + a x_4 + b x_5$

$(x_1 - x_4) + (\frac{4}{5}x_4 + \frac{3}{5}x_5) = \frac{9}{5}$

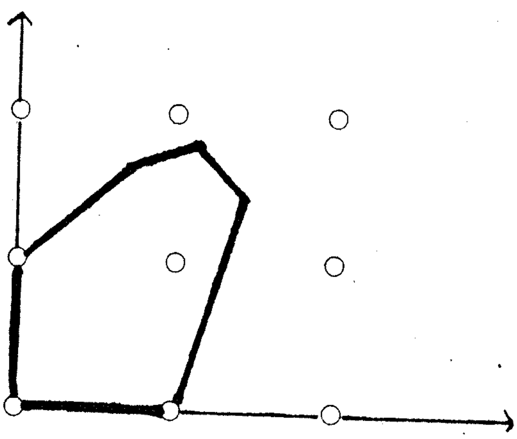
$x_1 - x_4 \le 1$

$-x_1 + 3x_2 \le 1$

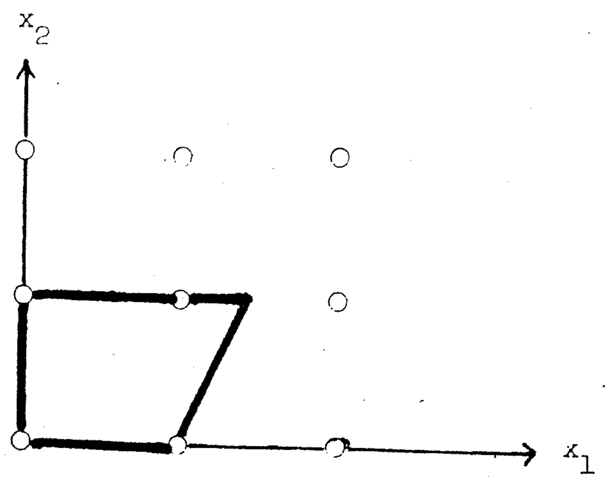
validos: $x_2 \le 2$



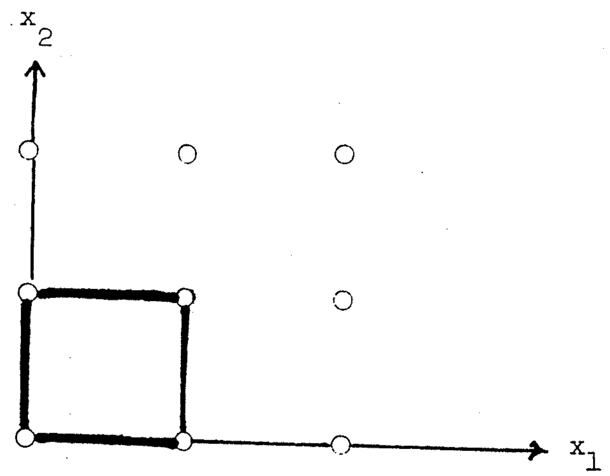
After the second cut



After the third cut



After the fourth cut



After the fifth cut

Finally, let us touch upon the question of "finite termination": it is conceivable that the algorithm, as we have described it so far, might keep on producing cuts in an endless sequence and never terminate. (However, the author must confess that he does not know any example where this actually happens.) Gomory has shown how to avoid such a misfortune by careful choices of the source rows [ ]. We shall not reproduce his proof.

This section ends on a sour note. True, the cutting plane algorithm always terminates. However, the sequence of cuts that is creates is usually very long, often too long to handle. Somewhat paradoxically, quite unsophisticated methods for solving ILP problems often perform better. Even so, there is no known algorithm for solving the ILP problems whose efficiency comes anywhere near to, say, that of the simplex method. In fact, there are results [ ], [ ] in the theory of "computational complexity" which strongly suggest that there is no "efficient" algorithm for solving the ILP problems. With the present level of computer technology, about 200 variables seems to be the upper bound on the size of ILP problems that can be solved within a reasonable time. (Of course, if the problems exhibit certain special structures then there may be efficient algorithms which exploit those special structures; such algorithms push the limits much farther.)